VE1.1: VECTORS: INTRODUCTION

Definition

A vector is a quantity that has a magnitude and a direction.

One example of a vector is velocity. The velocity of an object is determined by the magnitude (speed) and direction of travel. Other examples of vectors are force, displacement and acceleration.

A scalar is a quantity that has magnitude only. Mass, time and volume are all examples of scalar quantities.

Vectors in three-dimensional space are defined by three mutually perpendicular directions and will be denoted by lower case bold letters such as \( \mathbf{a}, \mathbf{b}, \mathbf{c} \).

A vector in the opposite direction from \( \mathbf{a} \) is denoted by \( -\mathbf{a} \).

Vectors can be added or subtracted graphically using the triangle rule.

Adding and subtracting vectors

Triangle Rule

To add vectors \( \mathbf{a} \) and \( \mathbf{b} \) shown above place the tail of vector \( \mathbf{b} \) at the head of vector \( \mathbf{a} \) (point Q).

The vector sum, \( \mathbf{a} + \mathbf{b} \), is the vector \( \overrightarrow{PR} \), from the tail of vector \( \mathbf{a} \) to the head of \( \mathbf{b} \).

To subtract \( \mathbf{b} \) from \( \mathbf{a} \), reverse the direction of \( \mathbf{b} \) to give \( -\mathbf{b} \) then add \( \mathbf{a} \) and \( -\mathbf{b} \).

\[
\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})
\]

Vector \( \overrightarrow{PR} \) is vector the vector \( \mathbf{a} - \mathbf{b} \).
**Components of a vector**

In the diagram below the vector \( \mathbf{r} \) is represented by \( \overrightarrow{OP} \) where \( P \) is the point \((x, y, z)\).

If \( i, j, k \) and \( \mathbf{k} \) are vectors of magnitude one parallel to the positive directions of the \( x \)-axis, \( y \)-axis and \( z \)-axis respectively, then:

- \( x \mathbf{i} \) is a vector of length \( x \) in the direction of the \( x \)-axis
- \( y \mathbf{j} \) is a vector of length \( y \) in the direction of the \( y \)-axis
- \( z \mathbf{k} \) is a vector of length \( z \) in the direction of the \( z \)-axis

\( \overrightarrow{OP} \) is then the vector \( x \mathbf{i} + y \mathbf{j} + z \mathbf{k} \).

\( x, y \) and \( z \) are called the **components** of the vector.

The notation \((x, y, z)\) will be used to denote the vector \((x \mathbf{i} + y \mathbf{j} + z \mathbf{k})\) as well as the co-ordinates of a point \( P(x, y, z) \). The context will determine the correct meaning.

Vectors may also be added or subtracted by adding or subtracting their corresponding components.

**Example**

If \( \mathbf{a} = (-3, 4, 2) \) and \( \mathbf{b} = (-1, -2, 3) \), find:

(i) \( \mathbf{a} + \mathbf{b} \)  
(ii) \( \mathbf{a} - \mathbf{b} \).

Adding or subtracting components

(i) \( \mathbf{a} + \mathbf{b} = (-3, 4, 2) + (-1, -2, 3) = (-3 + (-1), 4 + (-2), 2 + 3) = (-4, 2, 5) \)

Similarly (ii) \( \mathbf{a} - \mathbf{b} = (-3, 4, 2) - (-1, -2, 3) = (-3 - (-1), 4 - (-2), 2 - 3) = (-2, 6, -1) \)

See Exercise 1.
Directed line segment

The directed line segment, or geometric vector, $\overrightarrow{PQ}$, from $P(x_1, y_1, z_1)$ to $Q(x_2, y_2, z_2)$ is found by subtracting the co-ordinates of $P$ (the initial point) from the co-ordinates of $Q$ (the final point).

$$\overrightarrow{PQ} = ((x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)k)$$

Example

The directed line segment $\overrightarrow{PQ}$ is represented by the vector $2\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$, or $(2, 2, -2)$. Any other directed line segment with the same length and same direction as $\overrightarrow{PQ}$ is also represented by $2\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ or $(2, 2, -2)$.

The directed line segment $\overrightarrow{QP}$ has the same length as $\overrightarrow{PQ}$ but is in the opposite direction.

$$\overrightarrow{QP} = -\overrightarrow{PQ} = -(2\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}) = -2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k} \text{ or } (-2, -2, 2)$$

Position vector

The position vector of any point is the directed line segment from the origin $O$ $(0,0,0)$ to that point and is given by the co-ordinates of the point.

The position vector of $P(3, 4, 1)$ is $3\mathbf{i} + 4\mathbf{j} + \mathbf{k}$, or $(3, 4, 1)$.

See Exercise 2.
Magnitude of a vector

If \( \mathbf{a} = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k} \) the length or magnitude of \( \mathbf{a} \) is written \( |\mathbf{a}| \) or ‘a' and is evaluated as:

\[
|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}
\]

Example

The length of the vector \( 2\mathbf{i} + 3\mathbf{j} - 5\mathbf{k} \) equals \( \sqrt{2^2 + 3^2 + (-5)^2} = \sqrt{38} \).

\( a_1, a_2, a_3 \) are referred to as the components of vector \( \mathbf{a} \).

Unit vector

Any vector with a magnitude of one is called a unit vector. If \( \mathbf{a} \) is any vector then a unit vector parallel to \( \mathbf{a} \) is written \( \hat{\mathbf{a}} \) (\( \mathbf{a} \) “hat”). The “hat” symbolises a unit vector.

The unit vector \( \hat{\mathbf{a}} \) equals vector \( \mathbf{a} \) divided by its magnitude \( |\mathbf{a}| \)

\[
\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|}
\]

Or \( \mathbf{a} = |\mathbf{a}| \hat{\mathbf{a}} \)

Of particular importance are unit vectors \( \mathbf{i}, \mathbf{j} \) and \( \mathbf{k} \), parallel to the \( x \)-, \( y \)-, and \( z \)-axes respectively.

Examples

1. If \( \overrightarrow{PQ} \) is the line \( 2\mathbf{i} - 5\mathbf{j} + \mathbf{k} \) find a unit vector parallel to \( \overrightarrow{PQ} \)

\[
|\overrightarrow{PQ}| = \sqrt{2^2 + (-5)^2 + 1^2} = \sqrt{30}
\]

A unit vector parallel to \( \overrightarrow{PQ} \) is \( \frac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|} = \frac{\overrightarrow{PQ}}{\sqrt{30}} \)

\( \overrightarrow{PQ} = \frac{1}{\sqrt{30}} (2\mathbf{i} - 5\mathbf{j} + \mathbf{k}) \)

2. If \( \mathbf{a} = (1, 2, 3) \) a unit vector parallel to \( \mathbf{a} \) is:

\[
\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{(1, 2, 3)}{\sqrt{1^2 + 2^2 + 3^2}} = \frac{1}{\sqrt{14}} (1, 2, 3)
\]

See Exercise 3.
Multiplication by a scalar

To multiply vector \( \mathbf{a} = a_i \mathbf{i} + a_j \mathbf{j} + a_k \mathbf{k} \) by a scalar, \( m \), multiply each component of \( \mathbf{a} \) by \( m \).

\[
ma = ma_i \mathbf{i} + ma_j \mathbf{j} + ma_k \mathbf{k}
\]

The result is a vector of length \( m \times |\mathbf{a}| \).

If \( m > 0 \) the resultant vector is in the same direction as \( \mathbf{a} \).

If \( m < 0 \) the resultant vector is in the opposite direction from \( \mathbf{a} \).

Two vectors \( \mathbf{a} \) and \( \mathbf{b} \) are said to be parallel if and only if \( \mathbf{a} = k \mathbf{b} \) where \( k \) is a real constant.

Example

Multiply \( \mathbf{a} = (3\mathbf{i} + \mathbf{j} - 2\mathbf{k}) \) by 7 and show \( |7\mathbf{a}| = 7|\mathbf{a}| \)

\[
7\mathbf{a} = 7(3\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = 21\mathbf{i} + 7\mathbf{j} - 14\mathbf{k}.
\]

\[
|7\mathbf{a}| = \sqrt{21^2 + 7^2 + (-14)^2} = \sqrt{686} = 7\sqrt{14}
\]

The magnitude of \( \mathbf{a} \), \( |\mathbf{a}| \) is \( \sqrt{3^2 + 1^2 + (-2)^2} = \sqrt{14} \)

\[
7|\mathbf{a}| = 7\sqrt{14}
\]

therefore \( |7\mathbf{a}| = 7|\mathbf{a}| = 7\sqrt{14} \)

See Exercise 4.
Exercises

Exercise 1
Given \(a = (2,1,1), \ b = (1,3,-3)\) and \(c = (0,3,-2)\) find:

(a) \(a + b\), (b) \(a + c\), (c) \(c - b\), (d) \(a - b\)

Exercise 2
(a) Given the points \(A(3, 0, 4), \ B(-2, 4, 3), \) and \(C(1,-5, 0)\), find:

(i) \(\overrightarrow{AB}\) \(\overrightarrow{AC}\) \(\overrightarrow{CB}\)
(ii) \(\overrightarrow{BC}\) \(\overrightarrow{CA}\)

Compare your answers to (ii) and (v), and also to (iii) and (iv). What do you notice?

(b) What are the position vectors of \(A, B\) and \(C\).

Exercise 3
(a) Find the length of the vectors

(i) \((3, -1, -1)\), (ii) \((0, 2, 4)\), (iii) \((0, -2, 0)\)

(b) Given \(A(3, 0, 4)\) \(B(0, 4, 3)\) and \(C(1,-5, 0)\), find unit vectors parallel to (i) \(\overrightarrow{BA}\) \(\overrightarrow{CB}\) \(\overrightarrow{AC}\)

Exercise 4
(a) Find the following

(i) \(3 (i + 3j - 5k)\) (ii) \(-4 ( j - 3k)\)

(b) If \(a = (2, -2, 1), \ b = (0, 1, 1)\) and \(c = (-1, 3, -2)\) find:

(i) \(2a + 3b\) (ii) \(3a - 2b\) (iii) \(2a - b + 2c\) (iv) a unit vector parallel to \(2a - b\)

(c) Write down a vector three times the length of \((6i + 2j - 5k)\) and in the opposite direction.
Answers
Exercise 1
(a) (3, 4, −2)  (b) (2, 4, −1)  (c) (−1, 0, 1)  (d) (1, −2, 4)

Exercise 2
(a) (i) (-5, 4, -1)  (ii) (-2, -5, -4)  (iii) (-3, 9, 3)  (iv) (3, -9, -3)
   (v) (2, 5, 4)

\[ \overrightarrow{AC} = -\overrightarrow{CA} \quad \overrightarrow{BC} = -\overrightarrow{CB} \]

(b) \( \overrightarrow{OA} = 3\mathbf{i} + 4\mathbf{k} \), \( \overrightarrow{OB} = -2\mathbf{i} + 4\mathbf{j} + 3\mathbf{k} \), \( \overrightarrow{OC} = \mathbf{i} - 5\mathbf{j} \)

Exercise 3
(a) (i) \( \sqrt{11} \)  (ii) \( 2\sqrt{5} \)  (iii) 2

(b) (i) \( \frac{3}{\sqrt{26}} \)  (ii) \( \frac{-1}{\sqrt{91}} \)  (iii) \( \frac{-2}{\sqrt{45}} = \frac{-2}{3\sqrt{5}} \)

Exercise 4
(a) (i) (3\mathbf{i} + 6\mathbf{j} - 15\mathbf{k})  (ii) (-4\mathbf{j} + 12\mathbf{k})

(b) (i) (4, −1,5)  (ii) (6, −8,1)  (iii) (2,1, −3)  (iv) \( \frac{4}{\sqrt{42}} \)

(c) (-18\mathbf{i} − 6\mathbf{j} + 15\mathbf{k})