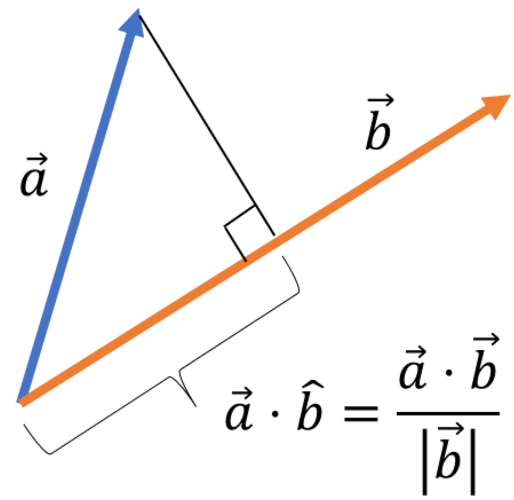
V5 Projection of Vectors



There are two types of vector projection:

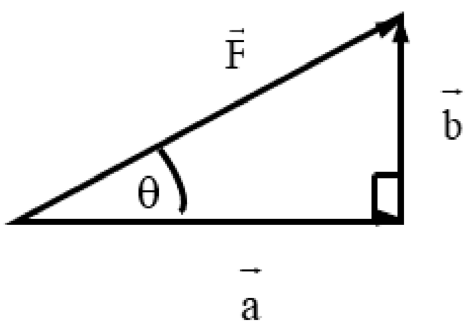
1. Scalar projection

2. Vector projection

For scalar projection, we calculate the length (a scalar quantity) of a vector in a particular direction.

For vector projection we calculate the vector component of a vector in a given direction.

Often, in Physics, Engineering and Mathematics courses you are asked to resolve a vector into two component vectors that are perpendicular to one another. As an example, in the diagram below a vector $\vec{a}$ is the projection of $\vec{F}$ in the horizontal direction while $\vec{b}$ is the projection of $\vec{F}$ in the vertical direction.

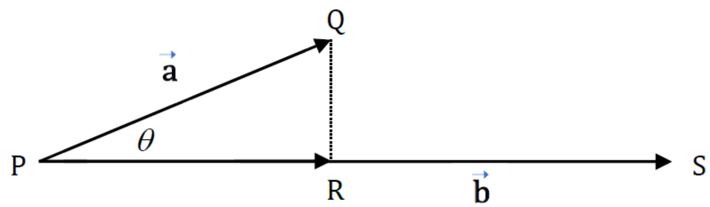


You can project a vector in any direction, not only horizontally and vertically.

This module discusses both scalar and vector projections.

Scalar Projection

Consider the following diagram:



Let $\overrightarrow{PQ} = \vec{a}$ and $\overrightarrow{PS} = \vec{b}$. The scalar projection of the vector $\vec{a}$ in the direction of vector $\vec{b}$ is the length of the straight line PR or $\left|\overrightarrow{PR}\right|$ where [1]

$$\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\left|\overrightarrow{PR}\right|}{|\vec{a}|}.$$

Rearranging gives,

$$\left|\overrightarrow{PR}\right| = |\vec{a}|\cos(\theta). \qquad (1)$$

This may be written in terms of the dot product [2].

We know

$$\vec{a}\cdot\vec{b} = |\vec{a}|\left|\vec{b}\right|\cos(\theta)$$

and so

$$\cos(\theta) = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\left|\vec{b}\right|}.$$

Substituting this expression for $\cos(\theta)$ into eqn(1) above gives:

$$\begin{aligned}\left|\overrightarrow{PR}\right| &= |\vec{a}|\cos(\theta)\\ \left|\overrightarrow{PR}\right| &= |\vec{a}|\,\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot\left|\vec{b}\right|}\\ &= \frac{\vec{a}\cdot b}{\left|\vec{b}\right|}\\ &= \vec{a}\cdot\frac{\vec{b}}{\left|\vec{b}\right|}\\ &= \vec{a}\cdot\hat{b}\end{aligned}$$

where $\hat{b} = \frac{\vec{b}}{|\vec{b}|}$ is the unit vector[3] in the direction of $\vec{b}$.

[1] Remember that the magnitude (or length) of a vector $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ is denoted by

$$|\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

For example, if $\vec{v} = 2\vec{i} - 3\vec{j} + \vec{k}$ then

$$\begin{aligned}|\vec{v}| &= \sqrt{2^2 + (-3)^2 + 1^2}\\ &= \sqrt{4+9+1}\\ &= \sqrt{14}\end{aligned}$$

[2] The dot product (or scalar product) of two vectors

$$\begin{aligned}\vec{a} &= a_1\vec{i} + a_2\vec{j} + a_3\vec{k}\\ \vec{b} &= b_1\vec{i} + b_2\vec{j} + b_3\vec{k}\end{aligned}$$

is defined to be

$$\vec{a}\cdot\vec{b} = |\vec{a}|\left|\vec{b}\right|\cos(\theta)$$

An alternate definition is

$$\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3.$$

[3] A unit vector for the vector $\vec{b}$ is denoted by $\hat{b}$ and is the vector $\vec{b}$ divided by it's length $\left|\vec{b}\right|$. That is

$$\hat{b} = \frac{\vec{b}}{\left|\vec{b}\right|}.$$

A unit vector is a vector of length one in the direction of the original vector.

The scalar projection of a vector $\vec{a}$ in the direction of vector $\vec{b}$ is given by:

$$\begin{aligned} \frac{\vec{a}\cdot\vec{b}}{\left|\vec{b}\right|} &= \vec{a}\cdot\hat{b} \\ &= |\vec{a}|\cos\theta \end{aligned}$$

Example

Find the scalar projection of the vector $\vec{a} = (2, 3, 1)$ in the direction of vector $\vec{b} = (5, -2, 2)$.

Solution:

The magnitude of $\vec{b}$ is

$$\begin{aligned} \left|\vec{b}\right| &= \sqrt{5^2 + (-2)^2 + 2^2} \\ &= \sqrt{25 + 4 + 4} \\ &= \sqrt{33} \end{aligned}$$

therefore

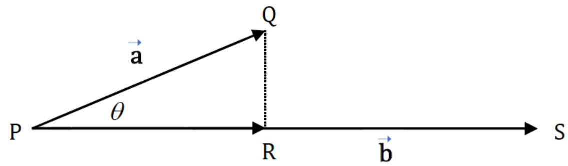
$$\begin{aligned} \hat{b} &= \frac{\vec{b}}{\left|\vec{b}\right|} \\ &= \frac{(5, -2, 2)}{\sqrt{33}} \\ &= \frac{1}{\sqrt{33}}(5, -2, 2) \end{aligned}$$

so the scalar projection of $\vec{a}$ in the direction of $\vec{b}$ is:

$$\begin{aligned} \vec{a}\cdot\hat{b} &= (2, 3, 1)\cdot\frac{(5, -2, 2)}{\sqrt{33}} \\ &= \frac{(2\times 5) + (3\times(-2)) + (1\times 2)}{\sqrt{33}} \\ &= \frac{10 - 6 + 2}{\sqrt{33}} \\ &= \frac{6}{\sqrt{33}}. \end{aligned}$$

Vector Projection

The vector projection of a vector $\vec{a}$ in the direction of vector $\vec{b}$ is a vector in the direction of $\vec{b}$ with magnitude equal to the length of the straight line PR or $\left|\overrightarrow{PR}\right|$ as shown below.



Therefore the vector projection of $\vec{a}$ in the direction of $\vec{b}$ is the scalar projection multiplied by a unit vector in the direction of $\vec{b}$.

The vector projection of vector $\vec{a}$ in the direction of vector $\vec{b}$ is:

$$\left(\vec{a}\cdot\hat{b}\right)\hat{b}=\frac{\left(\vec{a}\cdot\vec{b}\right)\vec{b}}{\left|\vec{b}\right|^2}$$

Example 1

Find the vector projection of vector $\vec{a}=(2,3,1)$ in the direction of vector $\vec{b}=(5,-2,2)$.

Solution:

The vector projection $\vec{a}$ in the direction of $\vec{b}$ equals:[4]

[4] Remember that

$$\begin{aligned}\left|\vec{b}\right| &= \sqrt{5^2+(-2)^2+2^2}\\ &= \sqrt{33}\end{aligned}$$

$$\begin{aligned}\left(\vec{a}\cdot\hat{b}\right)\hat{b} &= \frac{\left(\vec{a}\cdot\vec{b}\right)\vec{b}}{\left|\vec{b}\right|^2}\\ &= \frac{((2,3,1)\cdot(5,-2,2))\,(5,-2,2)}{\left(\sqrt{33}\right)^2}\\ &= \frac{(2\times5+3\times(-2)+1\times2)\,(5,-2,2)}{33}\\ &= \frac{(10-6+2)\,(5,-2,2)}{33}\\ &= \frac{(6)\,(5,-2,2)}{33}\end{aligned}$$

The vector projection of $\vec{a}$ in the direction of $\vec{b}$ is

$$\begin{aligned}\frac{6\,(5,-2,2)}{33} &= \frac{6}{33}\,(5,-2,2)\\ &= \frac{2}{11}\left(5\hat{\imath}-2\hat{\jmath}+2\hat{k}\right).\end{aligned}$$

Example 2

If $\vec{a}=(1,-2,2)$ and $\vec{b}=(5,-2,2)$ find:

(a) The scalar projection of $\vec{a}$ in the direction of $\vec{b}$.
(b) The vector projection of $\vec{a}$ in the direction of $\vec{b}$.

Solution:
(a) The scalar projection of $\vec{a}$ in the direction of $\vec{b}$ is $\vec{a} \cdot \hat{b}$.
If

$$\vec{b} = (5, -2, 2)$$

then

$$\left|\vec{b}\right| = \sqrt{33}$$

and

$$\begin{aligned} \hat{b} &= \frac{\vec{b}}{\left|\vec{b}\right|} \\ &= \frac{(5, -2, 2)}{\sqrt{33}} \\ &= \frac{1}{\sqrt{33}} (5, -2, 2)\,. \end{aligned}$$

Therefore

$$\begin{aligned} \vec{a} \cdot \hat{b} &= (1, -2, 2) \cdot \frac{(5, -2, 2)}{\sqrt{33}} \\ &= \frac{((1 \times 5) + ((-2) \times (-2)) + (2 \times 2))}{\sqrt{33}} \\ &= \frac{(5 + 4 + 4)}{\sqrt{33}} \\ &= \frac{13}{\sqrt{33}}. \end{aligned}$$

The scalar projection is $13/\sqrt{33}$.

(b) The vector projection of $\vec{a}$ in the direction of $\vec{b}$ is $\left(\vec{a} \cdot \hat{b}\right) \hat{b}$
from part (a)

$$\vec{a} \cdot \hat{b} = \frac{13}{\sqrt{33}}$$

so that the vector projection of $\vec{a}$ in the direction of $\vec{b}$ is

$$\begin{aligned} \left(\vec{a} \cdot \hat{b}\right) \hat{b} &= \left(\frac{13}{\sqrt{33}}\right) \frac{(5, -2, 2)}{\sqrt{33}} \\ &= \frac{13(5, -2, 2)}{33} \\ &= \frac{13}{33}(5, -2, 2) \\ &= \frac{13}{33} \left(5\hat{i} - 2\hat{j} + 2\hat{k}\right) \end{aligned}$$

The vector projection is $\frac{13}{33}(5, -2, 2)$ or $\frac{13}{33} \left(5\hat{i} - 2\hat{j} + 2\hat{k}\right)$

Exercise 1

For $\vec{a} = (2,3,1)$, $\vec{b} = (5,0,3)$, $\vec{c} = (0,0,3)$ and $\vec{d} = (-2,2,-1)$ find:

1. The scalar projection of $\vec{a}$ in the direction of $\vec{b}$.
   Answer: $\frac{13}{34}$
2. The vector projection of $\vec{a}$ in the direction of $\vec{b}$.
   Answer: $\frac{13}{34}\,(5,0,3)$
3. The scalar projection of $\vec{c}$ in the direction of $\vec{b}$.
   Answer: $\frac{9}{\sqrt{34}}$
4. The vector projection of $\vec{c}$ in the direction of $\vec{a}$.
   Answer: $\frac{3}{14}\,(2,3,1)$
5. The vector projection of $\vec{d}$ in the direction of $\vec{a}$.
   Answer: $\frac{1}{14}\,(2,3,1)$
6. The vector projection of $\vec{b}$ in the direction of $\vec{d}$.
   Answer: $\frac{-13}{9}\,(-2,2,-1)$

Exercise 2

For $\vec{a} = (2,0,-1)$, $\vec{b} = (3,5,6)$ find:

1. Find the scalar projection of $\vec{a}$ in the direction of $\vec{b}$.
   Answer: 0
2. What can you say about the relationship between $\vec{a}$ and $\vec{b}$?
   Answer: $\vec{a}$ and $\vec{b}$ are perpendicular. Remember $\cos 90^\circ = 0$ and see equation (1) above.