IL1.5: SURDS

Definition

A surd is an irrational number resulting from a radical expression that cannot be evaluated directly.

For example, $\sqrt{2}$, $\sqrt{3}$, $\frac{1}{\sqrt{8}}$, $\sqrt{20}$, $\sqrt{18}$, $\sqrt{20}$, $\sqrt{2}$, $\sqrt{9}$, $\sqrt{48}$, etc. are all surds.

but $\sqrt{1}$, $\sqrt{5}$, $\frac{1}{\sqrt{100}}$, $\sqrt{16}$, $\sqrt{64}$ are not.

Surds cannot be expressed exactly in the form $\frac{a}{b}$, and can only be approximated by a decimal.

eg: $\sqrt{2} \approx 1.414$

See Exercise 1

Simplifying Surds

If a surd has a square factor, ie 4, 9, 16, 25, 36…, then it can be simplified.

Examples

1. $\sqrt{200} = \sqrt{100 \times 2} = \sqrt{100} \times \sqrt{2} = 10\sqrt{2}$

2. $3\sqrt{48} = 3\sqrt{16 \times 3} = 3\sqrt{16} \times \sqrt{3} = 3 \times 4 \times \sqrt{3} = 12\sqrt{3}$

3. $\frac{\sqrt{24}}{4} = \frac{\sqrt{8 \times 3}}{4} = \frac{\sqrt{8} \times \sqrt{3}}{4} = \frac{2 \times \sqrt{3}}{4} = \frac{\sqrt{3}}{2}$

See Exercise 2
Addition and Subtraction

Only like surds can be added or subtracted

Examples

1. \(8\sqrt{5} - 3\sqrt{5} = 5\sqrt{5}\)

2. \(2\sqrt{3} + 5\sqrt{7} - 10\sqrt{3} = 5\sqrt{7} - 8\sqrt{3}\)

Sometimes it is necessary to simplify each surd first

3. \(\sqrt{18} - \sqrt{8} - \sqrt{20} = \sqrt{9\times 2} - \sqrt{4\times 2} - \sqrt{4\times 5}\)

\[= 3\sqrt{2} - 2\sqrt{2} - 2\sqrt{5}\]

\[= \sqrt{2} - 2\sqrt{5}\]

NB: \(\sqrt{5} + \sqrt{7} \neq \sqrt{12}\) because \(\sqrt{5}\) and \(\sqrt{7}\) are not ‘like’ surds.

See Exercise 3

Multiplication

To multiply surds the whole numbers are multiplied together, as are the numbers enclosed by the radical sign

\[a\sqrt{b} \times c\sqrt{d} = ac\sqrt{bd}\]

Examples:

1. \(\sqrt{2} \times \sqrt{3} = \sqrt{6}\)

2. \(3\sqrt{3} \times 4\sqrt{5} = 12\sqrt{15}\)

Sometimes it is possible to simplify after multiplication

3. \(2\sqrt{10} \times 7\sqrt{6} = 14\sqrt{60}\)

\[= 14\sqrt{4 \times 15}\]

\[= 14 \times 2\sqrt{15}\]

\[= 28\sqrt{15}\]

See Exercise 4
Expansion of Brackets

The usual algebraic rules for expansion of brackets apply to brackets containing surds

\[ a(b + c) = ab + ac \]

and

\[ (a + b)(c + d) = ac + bc + ad + bd \]

**Examples**

1. \[ \sqrt{2}(\sqrt{2} + 5) = 2 + 5\sqrt{2} \]

2. \[ 2\sqrt{3}(\sqrt{3} - 3\sqrt{2}) = 2\sqrt{9} - 6\sqrt{6} \]
   \[ = 6 - 6\sqrt{6} \]

3. \[ (7 - \sqrt{5})^2 = (7 - \sqrt{5})(7 - \sqrt{5}) \]
   \[ = 49 - 7\sqrt{5} - 7\sqrt{5} + 5 \]
   \[ = 54 - 14\sqrt{5} \]

4. \[ (\sqrt{6} - 4\sqrt{3})(2\sqrt{2} - 3\sqrt{5}) = 2\sqrt{12} - 8\sqrt{6} - 3\sqrt{30} + 12\sqrt{15} \]
   \[ = 2\times2\times\sqrt{3} - 8\sqrt{6} - 3\sqrt{30} + 12\sqrt{15} \]
   \[ = 4\sqrt{3} - 8\sqrt{6} - 3\sqrt{30} + 12\sqrt{15} \]

**See Exercise 5**

**Fractions containing surds**

Surds expressed as fractions may be simplified using

\[ \frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}} \]

**Examples**

1. \[ \frac{\sqrt{18}}{\sqrt{6}} = \frac{\sqrt{18}}{\sqrt{6}} = \sqrt{3} \]

2. \[ \frac{4}{\sqrt{2}} = \frac{\sqrt{16}}{\sqrt{2}} = \frac{\sqrt{16}}{\sqrt{2}} = \sqrt{8} \quad \text{or} \quad 2\sqrt{2} \]

**See Exercise 6**
Rationalizing surds

Rationalized surds are expressed with a rational denominator.

Examples

1. \[
\frac{2}{\sqrt{5}} = \frac{2}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} = \frac{2\sqrt{5}}{5}
\]

2. \[
\frac{\sqrt{5}}{3\sqrt{2}} = \frac{\sqrt{5}}{3\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{10}}{6}
\]

Conjugate surds

The pair of expressions \(\sqrt{a} + \sqrt{b}\) and \(\sqrt{a} - \sqrt{b}\) are called conjugate surds. Each is the conjugate of the other.

The product of two conjugate surds does NOT contain any surd term!

\[
(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = a - b
\]

Eg: \[
(\sqrt{10} - \sqrt{3})(\sqrt{10} + \sqrt{3}) = (\sqrt{10})^2 - (\sqrt{3})^2 = 10 - 3 = 7
\]

We make use of this property of conjugates to rationalize denominators of the form \(\sqrt{a} + \sqrt{b}\) and \(\sqrt{a} - \sqrt{b}\).

Example

\[
\frac{\sqrt{3}}{5 + \sqrt{2}} = \frac{\sqrt{3}}{5 + \sqrt{2}} \times \frac{5 - \sqrt{2}}{5 - \sqrt{2}} = \frac{\sqrt{3}(5 - \sqrt{2})}{25 - 2} = \frac{5\sqrt{3} - \sqrt{6}}{23}
\]

See Exercise 7
Operations on fractions that contain surds

When adding and subtracting fractions containing surds it is generally advisable to first rationalize each fraction:

Example

\[
\frac{2}{3\sqrt{2} + 1} + \frac{1}{3\sqrt{2} - \sqrt{2}} = \frac{2}{3\sqrt{2} + 1} \times \frac{3\sqrt{2} - 1}{3\sqrt{2} - 1} + \frac{1}{\sqrt{2}} \times \frac{\sqrt{3} + \sqrt{2}}{\sqrt{3} + \sqrt{2}}
\]

\[
= \frac{6\sqrt{2} - 2 + \sqrt{3} + \sqrt{2}}{18 - 1} + \frac{1}{3 - 2}
\]

\[
= \frac{6\sqrt{2} - 2}{17} + \frac{\sqrt{3} + \sqrt{2}}{1}
\]

\[
= \frac{6\sqrt{2} - 2 + 17\sqrt{3} + 17\sqrt{2}}{17}
\]

\[
= \frac{23\sqrt{2} - 2 + 17\sqrt{3}}{17}
\]

See Exercise 8

Exercises

Exercise 1
Decide whether the following radical expressions are rational or irrational and evaluate exactly if rational:

(a) \(\sqrt{25}\)  (b) \(\sqrt{2.25}\)  (c) \(\frac{1}{\sqrt{10}}\)  (d) \(\sqrt{64}\)  (e) \(\sqrt{12} \times 18\)

Exercise 2
Simplify

(a) \(\sqrt{20}\)  (b) \(\sqrt{48}\)  (c) \(\sqrt{500}\)  (d) \(2\sqrt{12}\)  (e) \(\frac{2}{\sqrt{20}}\)

Exercise 3
Simplify

(a) \(2\sqrt{3} + 5\sqrt{3}\)  (b) \(3\sqrt{7} + 5\sqrt{7} - 3\sqrt{7}\)

(c) \(\sqrt{2} + \sqrt{7} + 3\sqrt{2} - 4\sqrt{7}\)  (d) \(\sqrt{54} - \sqrt{24}\)

Exercise 4
Simplify

(a) \(\sqrt{3} \times \sqrt{10}\)  (b) \(2\sqrt{7} \times 5\sqrt{3}\)

(c) \(\sqrt{10} \times 3\sqrt{10}\)  (d) \(2\sqrt{8} \times 2\sqrt{50} \times \sqrt{2}\)

Exercise 5
Expand the brackets and simplify if possible

(a) \(\sqrt{2}(\sqrt{2} - 8)\)  (b) \((2 + \sqrt{3})(\sqrt{3} - 4)\)

(c) \((1 + \sqrt{10})^2\)  (d) \((\sqrt{11} + 3)(\sqrt{11} - 3)\)
Exercise 6
Simplify
(a) \( \frac{\sqrt{12}}{\sqrt{6}} \)  
(b) \( \frac{\sqrt{32}}{\sqrt{2}} \)  
(c) \( \frac{3\sqrt{28}}{\sqrt{7}} \)  
(d) \( \frac{4\sqrt{2} \times 3\sqrt{3}}{6\sqrt{6}} \)

Exercise 7
Express the following fractions with a rational denominator in simplest form:
1. (a) \( \frac{\sqrt{5}}{\sqrt{2}} \)  
   (b) \( \frac{1}{\sqrt{10}} \)  
   (c) \( \frac{2\sqrt{18}}{\sqrt{8}} \)
2. (a) \( \frac{2 + \sqrt{3}}{\sqrt{2}} \)  
   (b) \( \frac{1}{\sqrt{3} - \sqrt{2}} \)  
   (c) \( \frac{\sqrt{3} + 2}{\sqrt{3} - 2} \)

Exercise 8
Evaluate and express with a rational denominator \( \frac{2}{\sqrt{3} - 1} + \frac{3}{2 - \sqrt{3}} \)

Answers
Exercise 1
(a) 5 rational  (b) 1.5 rational  (c) irrational  (d) 4 rational  (e) irrational

Exercise 2
(a) \( 2\sqrt{5} \)  
   (b) \( 4\sqrt{5} \)  
   (c) \( 10\sqrt{5} \)  
   (d) \( 4\sqrt{3} \)  
   (e) \( \frac{1}{\sqrt{5}} \)

Exercise 3
(a) \( 7\sqrt{3} \)  
   (b) \( 5\sqrt{7} \)  
   (c) \( 4\sqrt{2} - 3\sqrt{7} \)  
   (d) \( \sqrt{6} \)

Exercise 4
(a) \( \sqrt{30} \)  
   (b) \( 10\sqrt{21} \)  
   (c) \( 30 \)  
   (d) \( 80\sqrt{2} \)

Exercise 5
(a) \( -2 - 8\sqrt{2} \)  
   (b) \( -5 - 2\sqrt{3} \)  
   (c) \( 11 + 2\sqrt{10} \)  
   (d) \( 2 \)

Exercise 6
(a) \( \sqrt{2} \)  
   (b) \( 4 \)  
   (c) \( 6 \)  
   (d) \( 2 \)

Exercise 7
1. (a) \( \frac{\sqrt{10}}{2} \)  
   (b) \( \frac{\sqrt{10}}{10} \)  
   (c) 3
2. (a) \( \frac{2\sqrt{2} + \sqrt{6}}{2} \)  
   (b) \( \sqrt{3} + \sqrt{2} \)  
   (c) \( -7 - 4\sqrt{3} \)

Exercise 8
\( 4\sqrt{3} + 7 \)