TR1.4: THE SINE RULE

The sine rule can be used to find angles and sides in any triangle (not just a right-angled triangle) when given:

- One side and any two angles
- Two sides and an angle opposite one of the given sides.

In the triangle ABC below:
- angles A, B, C, are the angles at the vertices A, B, C respectively
- a, b, c are the side lengths opposite the angles A, B, C respectively.

The sine rule states:
\[
\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad \text{or} \quad \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}
\]

Examples

1. In triangle PQR find:
   a) side length p
   b) side length q

   a) Side length p
   Use the sine rule in the form
   \[
   \frac{p}{\sin P} = \frac{q}{\sin Q} = \frac{r}{\sin R}
   \]
   The relevant part of the formula is
   \[
   \frac{p}{\sin 70^\circ} = \frac{15}{\sin 30^\circ}
   \]
   \[
   p = \frac{15 \times \sin 70^\circ}{\sin 30^\circ}
   \]
   \[
   p = 28.2 \text{ cm.}
   \]
b) Side length q

Angle Q is found using the fact that the sum of the three interior angles of a triangle add to 180°.

\[ \therefore Q = 180° - (70° + 30°) = 80° \]

From the sine rule

\[
\frac{p}{\sin P} = \frac{q}{\sin Q} \]

\[
\frac{28.2}{\sin 70°} = \frac{q}{\sin 80°} \]

\[
q = \frac{28.2 \times \sin 80°}{\sin 70°} \]

\[
q = 29.6\text{cm}. \]

2. In triangle ABC find:

a) angle C

b) angle A

c) side length a.

[Diagram of triangle ABC with labels A, B, C, c = 12 m, b = 20 m, angle B = 126°]

a) Angle C

Use the sine rule in the form

\[
\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \]

The relevant part of the formula is

\[
\frac{\sin B}{b} = \frac{\sin C}{c} \]

\[
\frac{\sin 126°}{20} = \frac{\sin C}{12} \]

\[
\sin C = \frac{12 \times \sin 126°}{20} \]

\[
\sin C = 0.485 \]

\[ C = \sin^{-1} 0.485 \]

\[ C = 29° \]

b) Angle A

\[ A = 180° - (126° + 29°) \]

\[ A = 25° \]

c) Side length a

Use the sine rule in the form

\[
\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \]

The relevant part of the formula is

\[
\frac{a}{\sin 25°} = \frac{b}{\sin 126°} \]

\[
a = \frac{20 \times \sin 25°}{\sin 126°} \]

\[ a = 10.4\text{m}. \]
3. Given a triangle ABC with angle \( A = 30^\circ \), adjacent side = 15 cm. and opposite side = 8cm. as shown below, find angle \( C \).

In this case there are two possible solutions. This is often called the ambiguous case of the sine rule.

Use the sine rule in the form \( \frac{\sin A}{a} = \frac{\sin C}{c} \)

\[
\frac{\sin 30^\circ}{8} = \frac{\sin C}{12}
\]

\[
\sin C = \frac{12 \times \sin 30^\circ}{8} = 0.75
\]

\( C = \sin^{-1} 0.75 \)

\( C = 48.6^\circ \) This solution gives the triangle ABC₁

Another solution to \( C = \sin^{-1} 0.75 \) is \( C = 131.4^\circ (180^\circ - 48.6^\circ) \) This solution gives triangle ABC₂

In any non-right-angled triangle, where two sides and the non-included angle are given check for the ambiguous case if:

- the angle is acute and
- the length of the side adjacent to the angle is greater than the length of the side opposite the angle.

**Exercises**

**Exercise 1**

For the following triangles find the unknown sides.

a)  

\[
\begin{array}{c}
\text{A} \\
30^\circ \\
\text{c} = ? \\
65^\circ \\
\text{b} = ? \\
\end{array}
\]

b)  

\[
\begin{array}{c}
\text{P} \\
45^\circ \\
r = ? \\
85^\circ \\
q = ? \\
\end{array}
\]

c)  

\[
\begin{array}{c}
\text{R} \\
130^\circ \\
t = ? \\
27^\circ \\
f = ? \\
\end{array}
\]

Exercise 2
For the following triangles find all unknown angles and sides.

a)  
\[
\begin{array}{c}
L \quad 8.5 \\
N \quad 5 \\
M \quad 80^\circ \\
\end{array}
\]

b)  
\[
\begin{array}{c}
A \quad 21 \\
B \quad 16 \\
C \quad 35^\circ \\
\end{array}
\]

c)  
\[
\begin{array}{c}
A \quad 4.7 \\
B \quad 6.4 \\
C \quad 65^\circ \\
\end{array}
\]

Answers.
Question 1
a) \(b = 18.1\), \(c = 19.9\)  
b) \(q = 53.5\), \(r = 41.2\)  
c) \(t = 9.1\), \(r = 20.2\).

Question 2
a) \(\angle L = 35.4^\circ\), \(\angle N = 64.6^\circ\), \(n = 7.8\)  
b) \(\angle C = 131.2^\circ\), \(\angle B = 13.8^\circ\), \(b = 6.7\)  
c) \(\angle B = 41.7^\circ\), \(\angle A = 73.3^\circ\), \(a = 6.8\).