Applications of Differentiation

DN1.10: RATES OF CHANGE

If there is a relationship between two or more variables, for example,

area and radius of a circle where \( A = \pi r^2 \)

or length of a side and volume of a cube where \( V = \ell^3 \)

then there will also be a relationship between the rates at which they change.

If \( y \) is a function of \( x \) ie \( y = f(x) \)

then \( f'(x) = \frac{dy}{dx} \) is the rate of change of \( y \) with respect to \( x \)

We can use differentiation to find the function that defines the rate of change between variables

\[ \frac{dA}{dr} = 2\pi r \] (differentiating with respect to \( r \))

and \[ \frac{dV}{d\ell} = 3\ell^2 \] (differentiating with respect to \( \ell \))

The chain rule can be used to find rates of change with respect to time:

\[ \frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt} \]

So that when \( A = \pi r^2 \):

\[ \frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt} \]

\[ = 2\pi r \frac{dr}{dt} \]

and when \( V = \ell^3 \):

\[ \frac{dV}{dt} = \frac{dV}{d\ell} \times \frac{d\ell}{dt} \]

\[ = 3\ell^2 \frac{d\ell}{dt} \]
Examples

1. A balloon has a small hole and its volume \( V \) (cm\(^3\)) at time \( t \) (sec) is \( V = 66 - 10t - 0.01t^2 \), \( t > 0 \)
Find the rate of change of volume after 10 seconds.

\[
\frac{dV}{dt} = -10 - 0.02t
\]
When \( t = 10 \),
\[
\frac{dV}{dt} = -10 - (0.02)(10)
= -10.2 \text{ cm}^3/\text{sec}
\]

2. The pressure \( P \), of a given mass of gas kept at constant temperature, and its
volume \( V \) are connected by the equation \( PV = 500 \).

Find \( \frac{dP}{dV} \) when \( V = 20 \).

\[
PV = 500
\]
\[
\text{ie. } P = \frac{500}{V}
\]
\[
\text{ie. } P = 500V^{-1}
\]

Then \( \frac{dP}{dV} = -500V^{-2} \)
When \( V = 20 \):
\[
\frac{dP}{dV} = -500(20)^{-2}
= -1.25
\]

3. Water is running out of a conical funnel at the rate of 5cm\(^3\)/s. The radius of the
funnel is 10 cm and the height is 20 cm. How fast is the water level dropping
when the water is 10 cm deep?

Let \( h \) be the depth, \( r \) the radius and \( V \) be the volume of the water at time \( t \)

Then \( \frac{dV}{dt} = -5 \) (since the rate is decreasing)

By similar triangles:
\[
\frac{r}{h} = \frac{10}{20}
\]
\[
r = \frac{h}{2}
\]

\[
V = \frac{1}{3} \pi r^2 h \text{ [formula for volume of a cone]}
\]
\[
= \frac{1}{12} \pi h^3
\]
\[
\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}
\]
\[
\frac{dV}{dt} = \frac{\pi h^2}{4} \times \frac{dh}{dt}
\]
When \( h = 10 \),

\[
-5 = \frac{\pi \times 100}{4} \times \frac{dh}{dt}
\]

\[
\frac{dh}{dt} = \frac{-1}{5\pi}
\]

The water level is dropping at a rate of \( \frac{1}{5\pi} \text{ cm/s} \)

**Exercises**

1. The radius of a spherical balloon is increasing at a rate of 3 cm/min. At what rate is the volume increasing when the radius is 5 cm?

2. If the displacement of an object from a starting point is given by \( s(t) = \sin(t) - 2\cos(t) \) find the velocity when \( t = 1 \).
   
   *Hint: \( v(t) = s'(t) \)*

3. The kinetic energy \( K \) of a body of mass \( m \) moving with speed \( v \) is given by \( K = \frac{1}{2}mv^2 \). Find a formula for the instantaneous rate of change of energy with respect to velocity for a body with a mass of 10kg.

4. The function \( n(t) = 200t - 100\sqrt{t} \) describes the spread of a virus where \( t \) is the number of days since the initial infection and \( n \) is the number of people infected. Find the rate at which \( n \) is increasing at the instant when \( t = 4 \).

5. The profit \( P \) made from selling a certain item is related to the number sold \( x \) by the formula \( P(x) = 10000 - x^2 + 520x \). What is the rate of change of profit with respect to number sold when \( x = 260 \)? What is the significance of this answer?

6. If \( y = \left(x - \frac{1}{x}\right)^2 \) find \( \frac{dx}{dt} \) when \( x = 2 \), given \( \frac{dy}{dt} = 1 \).

7. A hollow right circular cone is held vertex downwards beneath a tap leaking at the rate of 2 cm³/s. Find the rate of rise of water level when the depth is 6 cm given that the height of the cone is 18 cm and its radius 12 cm.

8. The area of a circle is increasing at the rate of 3 cm²/s. Find the rate of change of the circumference when the radius is 2 cm.

**Answers**

1. \( 300\pi \approx 942 \text{ cm}^3/\text{min} \)
2. 2.22
3. \( \frac{dK}{dv} = 10v \)
4. 175
5. 0 which means this is the point at which profit is maximized.
6. \( \frac{4}{15} \text{ cm/s} \approx 0.04 \text{ cm/s} \)
7. \( \frac{1}{8\pi} \text{ cm/s} \approx 0.04 \text{ cm/s} \)
8. 1.5 cm/s