SUMMER

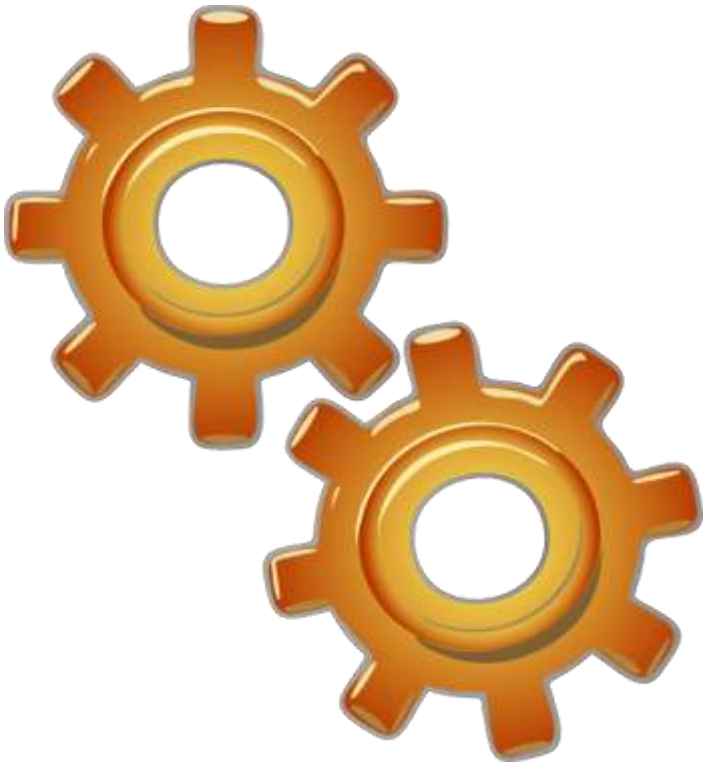
KNOWHOW

STUDY AND LEARNING CENTRE

RMIT UNIVERSITY

PHYSICS 2

AN ADVANCED PHYSICS KNOWHOW PROGRAM THAT BUILDS ON THE PHYSICS 1 WORKSHOP AND PREPARES STUDENTS FOR ENGINEERING AND APPLIED SCIENCE COURSES.



Contents

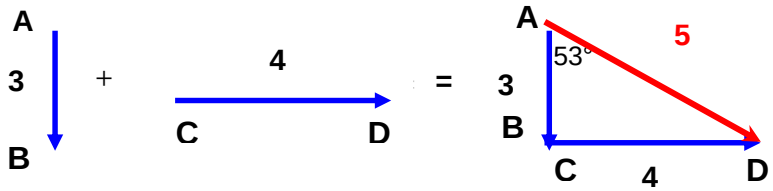
VECTORS: REVISION of FUNDAMENTALS

Most physics quantities can be classified into one of two groups. These groups are termed ***vector*** and ***scalar*** quantities. Scalar quantities are completely defined by a magnitude and the relevant unit, eg 10 seconds. Vector quantities, on the other hand, require magnitude and direction together with the relevant unit, eg. 10 kmh^{-1} due North.

Addition of vectors

Vectors are depicted diagrammatically by an arrow, whose length varies with the magnitude of the vector which points in the appropriate direction. A negative vector is simply a positive vector pointing the other way.

Example 3m/s South (**AB**) + 4 m/s East (**CD**). Use 'head-to-tail' rule to find resultant vector.

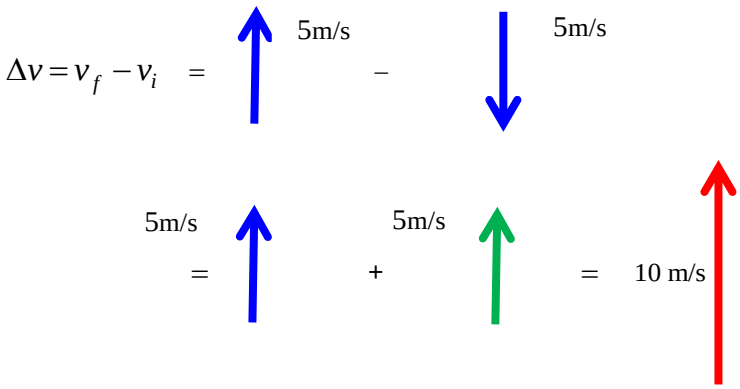


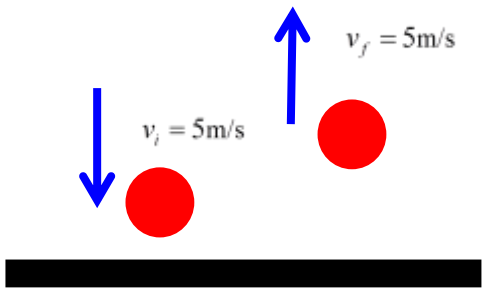
= 5 m/s South 53° East (127º T)
Use Pythagoras and trigonometry. This is sometimes called the Triangle Rule

Subtraction of vectors

When subtracting or finding the change in a vector, the initial value is taken away from the final value. The change in velocity Δv for example is given by $\Delta \mathbf{v} = \mathbf{v}_f - \mathbf{v}_i$

Example
A golf ball is dropped onto a concrete floor and strikes the floor with a speed of 5m/s. It then rebounds with a speed of 5m/s. What is the change in velocity of the ball?

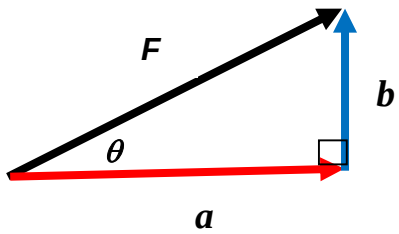




Note: Since velocity is a vector, vector subtraction is carried out.
Answer: 10 m/s UP

Resolution of vectors

A force ***F*** can be imagined as having two parts, or components, **a** and **b** that are at right angles to each other: **a** is called the horizontal component, and **b** the vertical component. The vector triangle showing the force and its component vectors is shown below.

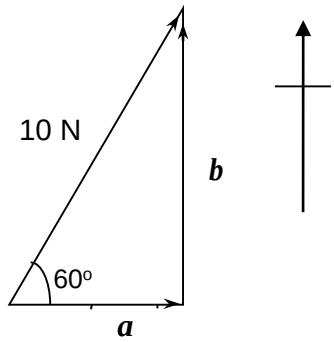


can be seen that

$$\sin\theta = \frac{b}{F} \quad \text{so} \quad b = F\sin\theta$$

$$\cos\theta = \frac{a}{F} \quad \text{so} \quad a = F\cos\theta$$

Example: Calculate the rectangular components of a force of 10 N that acts in a direction of North 30° East (030ºT)
The force and its rectangular components are shown. One component, ***a***, is acting to the east, the other, ***b***, to the north.

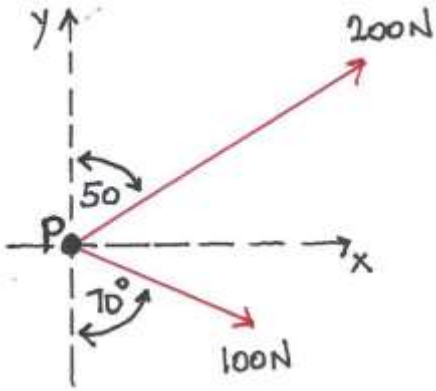


$$\boldsymbol{a} = \boldsymbol{F}\cos\theta \qquad \boldsymbol{b} = \boldsymbol{F}\sin\theta$$

$$= 10\cos 60° \qquad = 10\sin 60°$$
$$= 5\text{N} \qquad = 8.7\text{N}$$

Parallelogram resolution of vectors

If two vectors are represented by two adjacent sides of a parallelogram then the diagonal of the parallelogram will be equal to the resultant of these two vectors



Example: Two forces of 200N and 100N are acting at point P. See diagram. Calculate the magnitude and direction of the resultant force with respect to the *x*-axis.

Solution

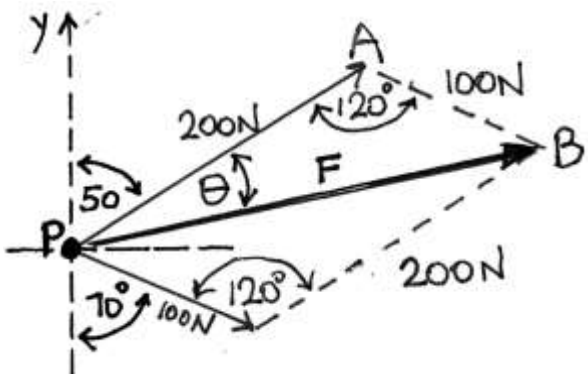
Draw a parallelogram using the 200N and 100N forces. See below. Draw the resultant force ***F*** which extends from P to the opposite corner of the parallelogram B. Use geometry to find the missing angles and substitute into the Cosine Rule:

$$a^2 = b^2 + c^2 - 2bc\,\text{Cos}\,A$$

$$a = \sqrt{b^2 + c^2 - 2bc\,\text{Cos}\,A}$$

$$= \sqrt{100^2 + 200^2 - 2(100)(200)\text{Cos}120}$$

$$\boldsymbol{F} = 264N$$



Use the Sine Rule to calculate θ.

$$\frac{\sin\theta}{100} = \frac{\sin 120}{264}$$

$$\theta = \sin^{1}\left(\frac{100 \times \sin 120}{264}\right)$$

$$= 19°$$

Hence the direction of ***F*** with respect to the x-axis is $90 - (50 + 19) = 21°$

Cosine law:
$C = \sqrt{A^2 + B^2 - 2AB\cos c}$
Sine law:
$\frac{A}{\sin a} = \frac{B}{\sin b} = \frac{C}{\sin c}$

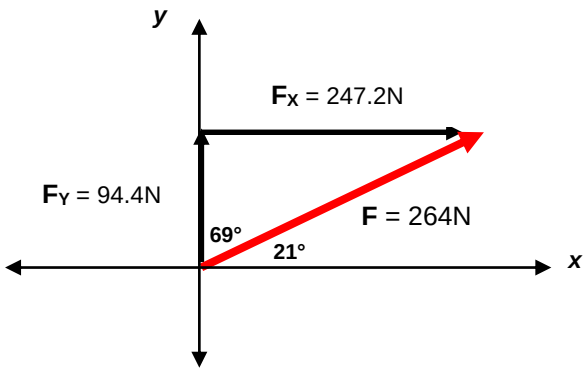
Alternatively …
Summing vertical and horizontal components

$\xrightarrow{+}$ $\sum F_X = 200\sin 50° + 100\sin 70° = +247.2\text{N}$

$+\uparrow$ $\sum F_Y = 200\cos 50° - 100\cos 70° = +94.4\text{N}$

Since the resultant force **F** = $\mathbf{F_X}$ + $\mathbf{F_Y}$,
and using Pythagoras and trigonometry:

F = 264N in a direction of 21° to the *x*-axis

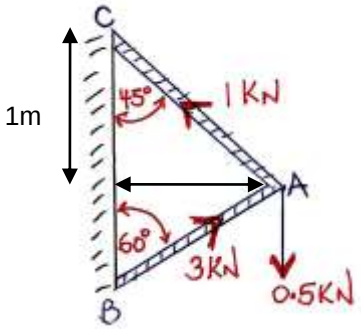


Exercise

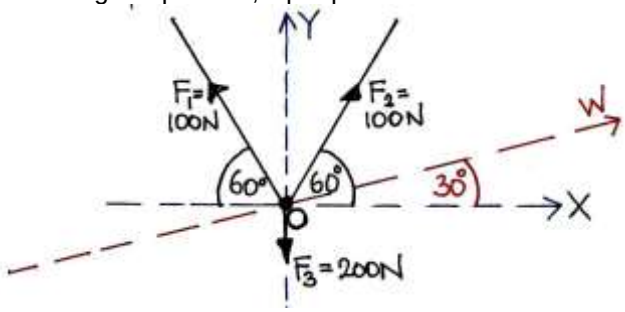
Answers to 1 decimal place. Take g = 9.81m/s^2

1 (a) To 60 km East add 25 km East. (b) To 60 km East add 25 km West.
2 (a) To 30 km West add 40 km North, (b) To 30 km West add 30 km South.
3 A ship sails 40km North East, and then changes direction and sails 40km South East. Find the displacement of the ship (vector sum).
4 Two forces, 100 Newtons North and 100 Newtons East, are acting away from the same point and are at 90º to each other. Find their vector sum.
5 A plane is flying due North at 300 km h^{-1} and a westerly wind of 25 km h^{-1} is blowing it off course. Calculate the true speed and direction of the plane.
6 Carry out the following vector subtractions: (a) 24 m N minus 18 m North, (b) 48 m N minus 22 m South, (c) 13.8 ms^{-1} East minus 9.4 ms^{-1} North.
7 A ball is thrown in an Easterly direction and strikes a wall at 12.5 ms^{-1}. If it rebounds at the same speed, what is its (a) change in speed, (b) change in velocity.
8 From 14.14 kmh^{-1} East subtract 14.14 kmh^{-1} North.
9 What is the change in velocity when an aeroplane travelling at 250 kmh^{-1} West alters its course to 350 kmh^{-1} North?
10 What is the northerly component of a wind blowing at 15ms^{-1} from the South East?
11 What is the magnitude of the vector whose components are 5N horizontally and 12N vertically?
12 What is the vector whose components are 3N North East and 4N South East?
13 What are the vertical and horizontal components of a force of 25N acting at 30 º to the horizontal?
14 A plane taking off leaves the runway at 32º to the horizontal, travelling at 180kmh^{-1}. How long will it take to climb to an altitude of 1000m?
15 A boat needs to get travel directly across from one side of a river to the other. Its speed in still water is 10km/h. If the river's current is 5km/h, (a) what upstream angle relative to the river bank must the navigator point the boat to do this, and (b) how wide (to the nearest metre) is the river if it takes 2 minutes to cross it?
16 A plane is travelling at 500km/h in a direction of 30ºT. It encounters a cross-wind blowing at a speed of 100km/h in a direction of 250ºT. What is the magnitude and direction of the plane's resultant velocity?
17 A plane is travelling at 150m/s and needs to get to its destination due north of its starting position. A cross wind of 50m/s is blowing in a direction 110ºT. What direction must the plane point the plane to get to its destination

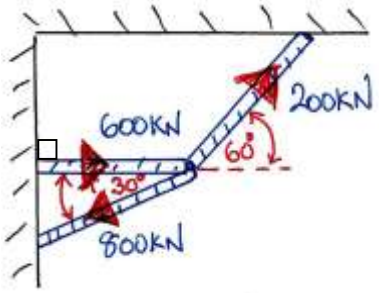
18. Calculate, by summing the vertical and horizontal components, the resultant force (magnitude in kN and direction measured counter clockwise from x-axis) acting on the 2 members AC, AB, and the 0.5kN weight acting at A below.



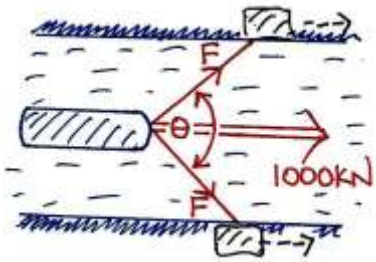
19. Refer to diagram below. Resolve
(a) F_1 into components along the x- and y-axes.
(b) F_2 into a component along the w-axis.
(c) F_3 into components along the x, w and y-axes.
Note. sign convention is:
To the right = positive; Up = positive



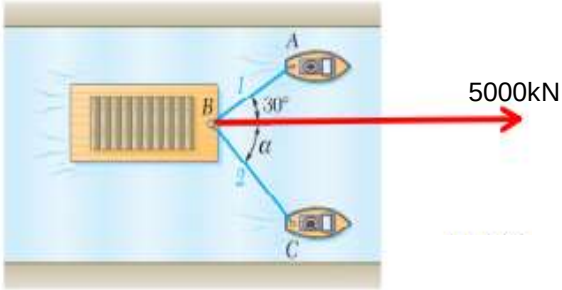
20. Calculate, by summing the vertical and horizontal components, the resultant force (magnitude and direction measure counter clockwise from x-axis) acting on the 3 members shown below. Give answers to 2 decimal places.



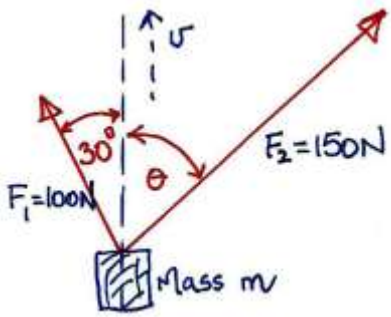
21. Two tow ropes of force F each and acting at an angle θ apart, where θ = 90°, are pulling a barge along a canal parallel to the canal bank. See diagram below. The resultant force is 1000kN. Calculate the magnitude of force F.



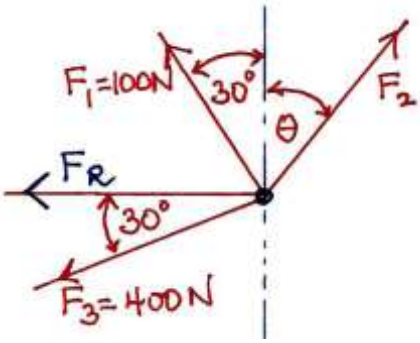
22. Two tow ropes of tension T_1 and T_2 (1 and 2 in the diagram at right), are pulling a barge along a canal parallel to the canal bank with a force of 5000kN.
(a) If α = 45° find T_1 and T_2
(b) Find α for which the value of T_2 is a minimum.



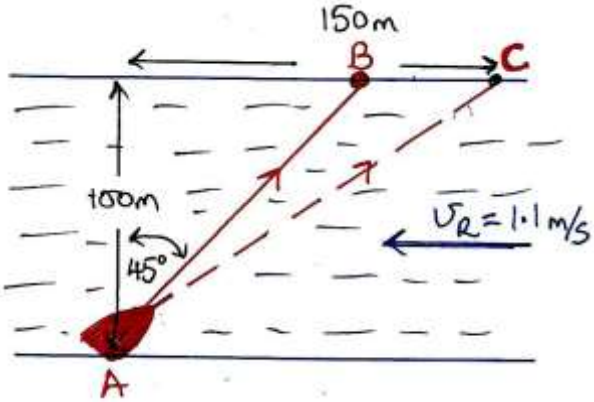
23. (a) If mass m is to be lifted vertically up from the ground, at what angle θ must the 150N force act?
(b) It is required that F_2 must have a minimum magnitude to lift mass m. What is the magnitude of F_2 and
(c) what is angle θ?



24. F_R is the resultant force of F_1, F_2, and F_3. See diagram at right. Calculate
(a) the value of F_R for which F_2 is a minimum.
(b) θ for this minimum value



25. A boat at point A needs to get to point B on the other side of a river. The river's current v_R is 1.1 km/h. The boat's navigator must point the boat at point C so that it gets to its destination.
(a) At what angle relative to the river bank must the boat's navigator point the boat?
(b) Calculate the boat's speed to enable it to get to point B?



Answers

1. (a) 85 km East, (b) 35 km East 2.(a) 50 km North 37º West (323ºT) (b) 42 km South West (225ºT)

3. 56.6 km East 4. 141.4 Newtons North East. 5. 301.0 kmh^{-1} North 5º East (005ºT).

6. (a) 6m North (b) 70 m North (c) 16.7 ms^{-1} S56E (or 124ºT) 7. (a) zero (b) 25 ms^{-1} West

8. 20 kmh^{-1} South East 9. 430 kmh^{-1} North 35° East (035ºT). 10. 10.6N 11. 13ms^{-1}

12. 5N 098 ºT 13. 12.5N vertical, 21.6N horizontal. 14. 37.8s. 15. (a) 60º relative to the river bank (b) 1039m; 16. 428.2km/h; 21ºT 17. 341.8ºT 18. 3.5kN, 39.4° 19. (a) -50N; 87N (b) 87N (c) 0; -100; 200

20. 1363.9kN at 102.5° measured counter clockwise from *x*-axis. 21. 707.1kN 22. T_1 = 3660kN, T_2= 2590kN (b) 60° 23. (a)19.5° (b) 50N (c) 60° 24. (a) 396.4N (b) 0° (c)396.4N 25. (a) 33.7º (b) 4.0m/s.

Additional Questions

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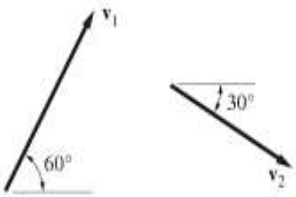


Fig. P1.22, P1.23

1.22 The magnitudes of the two velocity vectors are $v_1 = 3$ m/s and $v_2 = 2$ m/s. Determine their resultant $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$.

1.23 Determine the magnitudes of vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ so that their resultant is a horizontal vector of magnitude 4 m/s directed to the right.

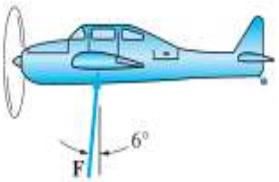


Fig. P1.24

1.24 The total aerodynamic force **F** acting on the airplane has a magnitude of 6250 lb. Resolve this force into vertical and horizontal components (called the lift and the drag, respectively).

1.25 Resolve the 200-lb force into components along (a) the x- and y-axes and (b) the x'- and y-axes.

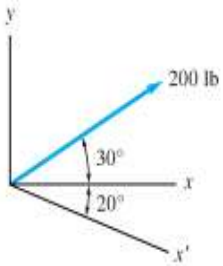
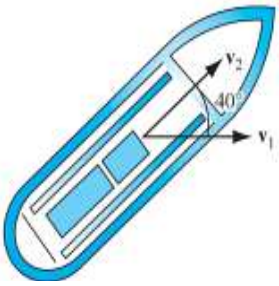


Fig. P1.25

1.26 The velocity vector of the boat has two components: $\mathbf{v}_1$ is the velocity of the water, and $\mathbf{v}_2$ is the velocity of the boat relative to the water. If $v_1 = 3$ mi/h and $v_2 = 5$ mi/h, determine the velocity vector of the boat.



1.27 The two tugboats apply the forces **P** and **Q** to the barge, where $P = 76$ kN and $Q = 52$ kN. Determine the resultant of **P** and **Q**.

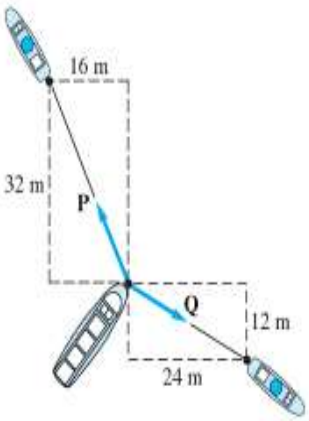


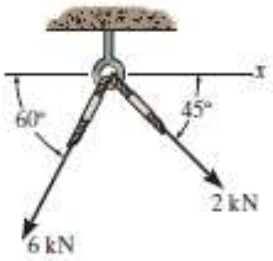
Fig. P1.27

Answers

1.22	3.61m/s at 26.3° above horizontal.	1.23	v_1 = 2m/s, v_2 = 3.5m/s
1.24	Lift = 6220lb ↑ Drag = 653lb →	1.25	(a) x-axis = 173lb, y-axis = 100lb
1.25	(b) x'-axis = 129lb, y'-axis = 153lb	1.26	7.55 mi/h at 15.2° above horizontal
1.27	46.5kN at 106° clockwise from the 9 o'clock position		

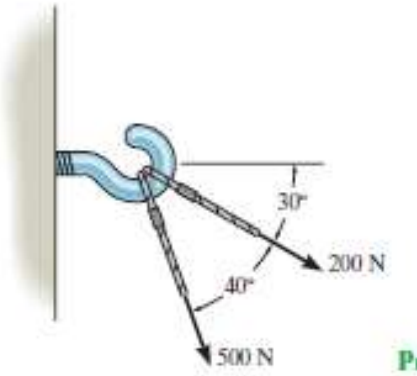
Calculate the resultant forces for F2-1, F2-2, F2-3 by summing the vertical and horizontal components.

F2–1. Determine the magnitude of the resultant force acting on the screw eye and its direction measured clockwise from the x axis.



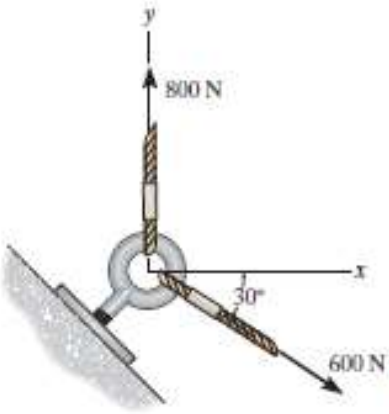
Prob. F2–1

F2–2. Two forces act on the hook. Determine the magnitude of the resultant force.



Prob. F2–2

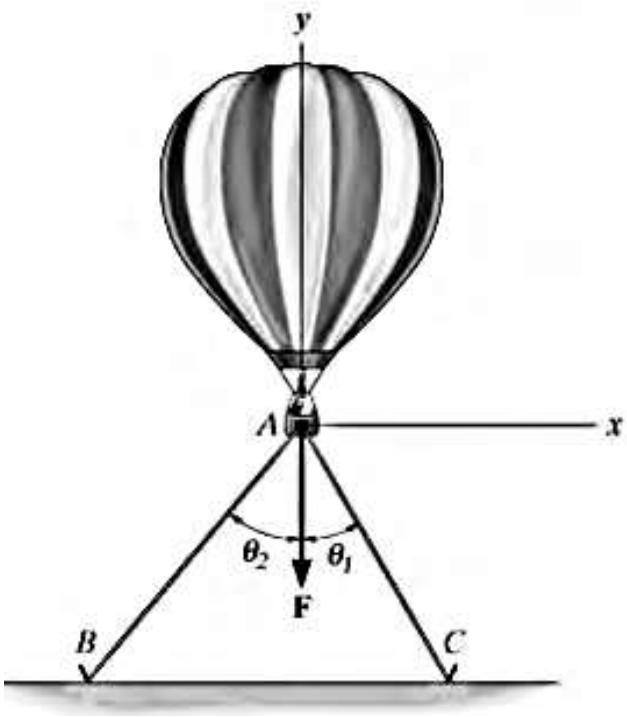
F2–3. Determine the magnitude of the resultant force and its direction measured counterclockwise from the positive x axis.



Prob. F2–3

F2.4

A resultant force **F** is necessary to hold the balloon in place. Calculate the magnitude of each of the ropes F_{AB} and F_{AC} by summing their vertical and horizontal components. Data: F = 350N, $\theta_1 = 30°$, $\theta_2 = 40°$.



Answers

F2.1 6.8kN
F2.2 665.7N
F2.3 1216.5N, 115°
F2.4 F_{AB} = 186N, F_{AC} = 239N

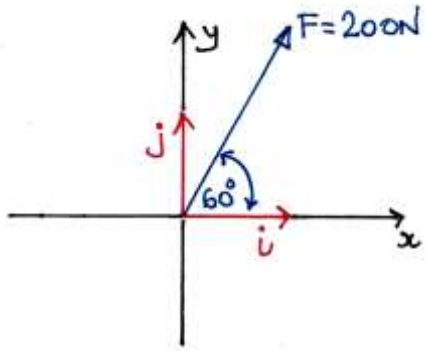
REPRESENTATION and ANALYSIS of 2- and 3-DIMENSIONAL FORCES

Unit vectors and vectors in component form

A unit vector is a vector acting in a given direction of magnitude 1.

In ***2-dimensional space***, unit vector ***i*** (*x*-axis) is at right angles to unit vector ***j*** (*y*-axis). See diagram.

For instance, vector ***A*** in 2-dimensional space can be represented by $\boldsymbol{A} = \boldsymbol{A}_x \boldsymbol{i} + \boldsymbol{A}_y \boldsymbol{j}$, where $\boldsymbol{A}_x$ and $\boldsymbol{A}_y$ are the components of vector ***A*** and ***i*** and ***j*** indicate the direction of the component vectors respectively.



Example

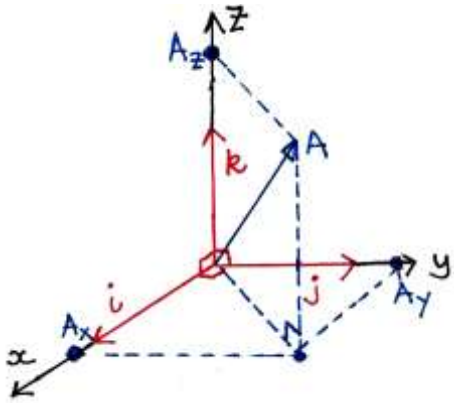
Force ***F*** shown in the diagram can be represented as a vector of magnitude 200N, in a direction of 60°above the *x*-axis.

Alternatively, ***F*** can be resolved into its *x*-and *y*-components as

$$\boldsymbol{F} = \boldsymbol{F}_x i + \boldsymbol{F}_y j \quad \text{or} \quad \boldsymbol{F} = (200\cos 60)i + (200\sin 60)\,j$$

$$= \left(100i + 100\sqrt{3}j\right)\text{N} \quad \text{or} \quad (100, 100\sqrt{3})\text{N}$$

In ***3-dimensional space***, each unit vector is orthogonal, ie vector ***i*** is at right angles to vectors ***j*** and ***k***, vector ***j*** is at right angles to vectors ***i*** and ***k*,** and vector ***k*** is at right angles to vectors ***i*** and ***j***. See diagram.



In terms of components, vector ***A*** can be represented by

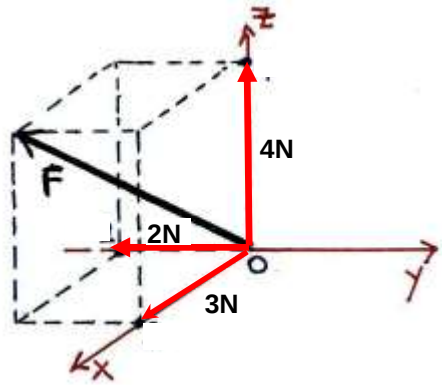
$$\boldsymbol{A} = \boldsymbol{A}_x i + \boldsymbol{A}_y j + \boldsymbol{A}_z k$$

Example

Force ***F*** shown in the diagram can be represented as

$$\boldsymbol{F} = \boldsymbol{F}_x i + \boldsymbol{F}_y j + \boldsymbol{F}_z k$$

$$= (3i - 2j + 4k)\,\boldsymbol{N} \quad \text{or} \quad (3, -2, 4)\,\boldsymbol{N}$$



Magnitude of a vector

Using Pythagoras' theorem, the magnitude (or length) of vector ***A*** in 2-dimensional space is

$$\boxed{|\boldsymbol{A}| = \sqrt{|\boldsymbol{A}_x|^2 + |\boldsymbol{A}_y|^2}}$$

Extending this to vectors in 3-dimensions

$$\boxed{|\boldsymbol{A}| = \sqrt{|\boldsymbol{A}_x|^2 + |\boldsymbol{A}_y|^2 + |\boldsymbol{A}_z|^2}}$$

Example

The magnitude of the force in 2 dimensions, $\boldsymbol{F} = 100i + 100\sqrt{3}j$ mentioned earlier is

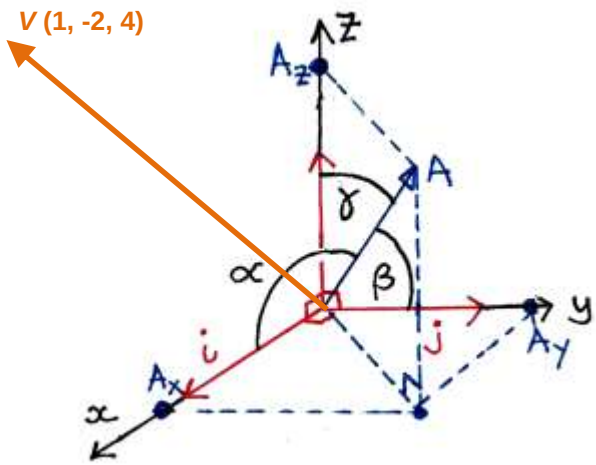
$|\boldsymbol{F}| = \sqrt{(100)^2 + (100\sqrt{3})^2} = 200$ N, as one would expect!

The magnitude of the force in 3 dimensions, $\boldsymbol{F} = 3i - 2j + 4k$, mentioned earlier is

$|\boldsymbol{F}| = \sqrt{(3)^2 + (-2)^2 + (4)^2} = \sqrt{29}$ N

Direction of a vector (direction cosines)

The direction of vector ***A*** is the angle between ***A*** and each of the coordinate axes *x*, *y* and *z*. See diagram. Using trigonometry, it can be seen that



$$\boxed{\cos\alpha = \frac{\boldsymbol{A}_x}{|\boldsymbol{A}|} \qquad \cos\beta = \frac{\boldsymbol{A}_y}{|\boldsymbol{A}|} \qquad \cos\gamma = \frac{\boldsymbol{A}_z}{|\boldsymbol{A}|}}$$

Since $\boldsymbol{A} = \boldsymbol{A}_x\boldsymbol{i} + \boldsymbol{A}_y\boldsymbol{j} + \boldsymbol{A}_z\boldsymbol{k}$ then

$$\boxed{\boldsymbol{A} = |\boldsymbol{A}|\cos\alpha\,\boldsymbol{i} + |\boldsymbol{A}|\cos\beta\,\boldsymbol{j} + |\boldsymbol{A}|\cos\gamma\,\boldsymbol{k}}$$

It can also be shown that

$$\boxed{\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1}$$

Example

Given vector ***V*** (1, -2, 4) shown above

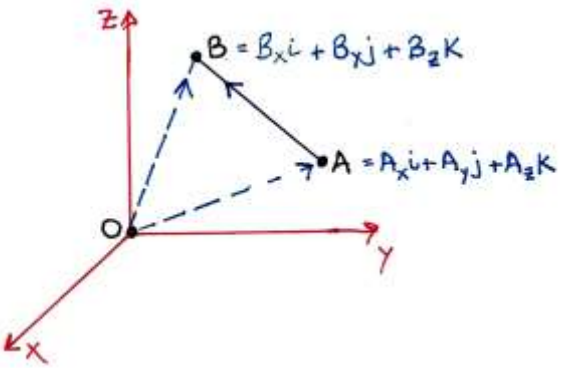
$$\alpha = \cos^{-1}\left(\frac{V_x}{|V|}\right) \qquad \beta = \cos^{-1}\left(\frac{V_y}{|V|}\right) \qquad \gamma = \cos^{-1}\left(\frac{V_z}{|V|}\right)$$

$$\alpha = \cos^{-1}\left(\frac{1}{\sqrt{21}}\right) \qquad \beta = \cos^{-1}\left(\frac{-2}{\sqrt{21}}\right) \qquad \gamma = \cos^{-1}\left(\frac{4}{\sqrt{21}}\right)$$

$$\alpha = 77° \qquad \beta = 116° \qquad \gamma = 29°$$

Position vectors

Vector ***AB*** is shown in the diagram. Using the 'head-to-tail' rule for resultant vectors



$$\boldsymbol{OB} = \boldsymbol{OA} + \boldsymbol{AB}$$

Hence

$$\begin{aligned}\boldsymbol{AB} &= \boldsymbol{OB} - \boldsymbol{OA}\\ &= \left(\boldsymbol{B}_x\boldsymbol{i} + \boldsymbol{B}_y\boldsymbol{j} + \boldsymbol{B}_z\boldsymbol{k}\right) - \left(\boldsymbol{A}_x\boldsymbol{i} + \boldsymbol{A}_y\boldsymbol{j} + \boldsymbol{A}_z\boldsymbol{k}\right)\\ &= \left(\boldsymbol{B}_x - \boldsymbol{A}_x\right)\boldsymbol{i} + \left(\boldsymbol{B}_y - \boldsymbol{B}_y\right)\boldsymbol{j} + \left(\boldsymbol{B}_z - \boldsymbol{A}_z\right)\boldsymbol{k}\end{aligned}$$

$\boldsymbol{OA}$ and $\boldsymbol{OB}$ are called position vectors as they give the position of a point relative to a fixed point, in this case the origin.

Example

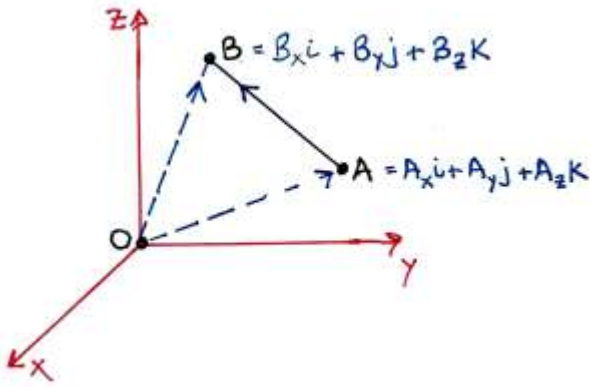
Referring to the diagram above, if $\boldsymbol{OA}$ and $\boldsymbol{OB}$ are position vectors where $\boldsymbol{OA} = (2i + 3j - 4k)$ m and $\boldsymbol{OB} = (2i - 5j + 7k)$ m, then

$$\boldsymbol{AB} = \boldsymbol{OB} - \boldsymbol{OA} = \left(2i - 5j + 7k\right) - \left(2i + 3j - 4k\right)$$

$$\boldsymbol{AB} = 0i - 8j + 11k = (-8j + 11k)\ \text{m}$$

Unit vector in a given direction

A unit vector $\hat{\boldsymbol{v}}_{AB}$ in the direction of ***AB*** is given by



$$\boxed{\hat{\boldsymbol{v}}_{AB} = \frac{\boldsymbol{AB}}{|\boldsymbol{AB}|} = \frac{(\boldsymbol{AB})_x}{|\boldsymbol{AB}|}i + \frac{(\boldsymbol{AB})_y}{|\boldsymbol{AB}|}j\frac{(\boldsymbol{AB})_z}{|\boldsymbol{AB}|}k}$$

where $|\boldsymbol{AB}|$ is the magnitude (or length) of vector ***AB***, or

$$AB = |\boldsymbol{AB}| = \sqrt{\left(|\boldsymbol{AB}|\right)_x^{\ 2} + \left(|\boldsymbol{AB}|\right)_y^{\ 2} + \left(|\boldsymbol{AB}|\right)_z^{\ 2}}$$

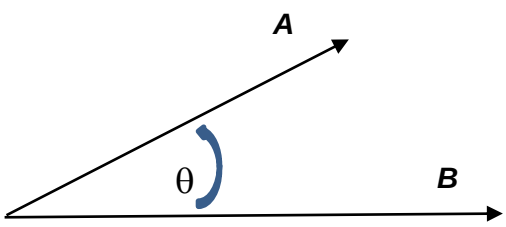
Example

Given $\boldsymbol{AB} = -8\boldsymbol{j} + 11\boldsymbol{k}$ from example above

$$\begin{aligned}\hat{\boldsymbol{v}}_{AB} &= \frac{\boldsymbol{AB}}{|\boldsymbol{AB}|} = \frac{(0,-8,11)}{\sqrt{0^2 + (-8)^2 + (11)^2}}\\ &= \frac{(0,-8,11)}{\sqrt{185}} = \frac{-8}{\sqrt{185}}j + \frac{11}{\sqrt{185}}k\end{aligned}$$

Scalar (Dot) product

The scalar product of two vectors ***A*** and ***B*** is



$$\boxed{\boldsymbol{A} \cdot \boldsymbol{B} = |\boldsymbol{A}||\boldsymbol{B}|\cos\theta}$$

where $|\boldsymbol{A}|$ and $|\boldsymbol{B}|$ are the magnitudes of the vectors and θ is the angle between the two vectors. See diagram above.

The scalar product is commonly used to find the angle between vectors.

If $\boldsymbol{A}\cdot\boldsymbol{B} = |\boldsymbol{A}||\boldsymbol{B}|\cos\theta, \quad \theta = \cos^{-1}\left(\dfrac{\boldsymbol{A}\cdot\boldsymbol{B}}{|\boldsymbol{A}||\boldsymbol{B}|}\right)$

If $\boldsymbol{A}\cdot\boldsymbol{B} = 0$ then $\boldsymbol{A}$ and $\boldsymbol{B}$ are perpendicular

In Cartesian form, where $\boldsymbol{A} = \boldsymbol{A}_x\boldsymbol{i} + \boldsymbol{A}_y\boldsymbol{j} + \boldsymbol{A}_z\boldsymbol{k}$ and $\boldsymbol{B} = \boldsymbol{B}_x\boldsymbol{i} + \boldsymbol{B}_y\boldsymbol{j} + \boldsymbol{B}_z\boldsymbol{k}$

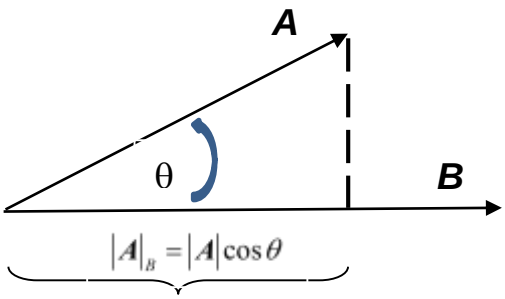
$$\begin{aligned}\boldsymbol{A}\cdot\boldsymbol{B} &= \left(\boldsymbol{A}_x\boldsymbol{i} + \boldsymbol{A}_y\boldsymbol{j} + \boldsymbol{A}_z\boldsymbol{k}\right)\cdot\left(\boldsymbol{B}_x\boldsymbol{i} + \boldsymbol{B}_y\boldsymbol{j} + \boldsymbol{B}_z\boldsymbol{k}\right)\\ &= \boldsymbol{A}_x\boldsymbol{B}_x + \boldsymbol{A}_y\boldsymbol{B}_y + \boldsymbol{A}_z\boldsymbol{B}_z\end{aligned}$$

Example

If $\boldsymbol{OA} = 2i + 3j - 4k$ and $\boldsymbol{OB} = 2i - 5j + 7k$

the scalar product $\boldsymbol{OA}\cdot\boldsymbol{OB}$ is

$$\begin{aligned}\boldsymbol{OA}\cdot\boldsymbol{OB} &= (2,3,-4)\cdot(2,-5,7)\\ &= (2\times 2) + (3\times -5) + (-4\times 7) = -39\end{aligned}$$



Scalar projection

The scalar projection (or component) of vector $\boldsymbol{A}$ on to $\boldsymbol{B}$ is given by

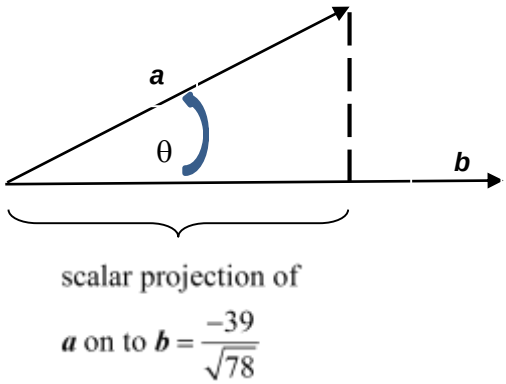
$$\boxed{|\boldsymbol{A}|_B = |\boldsymbol{A}|\cos\theta = \boldsymbol{A}\cdot\hat{\boldsymbol{B}}}$$

Example

If $\boldsymbol{OA} = 2i + 3j - 4k$ and $\boldsymbol{OB} = 2i - 5j + 7k$

Let $\boldsymbol{OA} = \boldsymbol{a}$ and $\boldsymbol{OB} = \boldsymbol{b}$ The scalar projection of $\boldsymbol{a}$ on to $\boldsymbol{b}$ is

$$\begin{aligned}\boldsymbol{a}\cdot\hat{\boldsymbol{b}} &= \boldsymbol{a}\cdot\frac{\boldsymbol{b}}{|\boldsymbol{b}|} = (2,3,-4)\cdot\frac{(2,-5,7)}{\sqrt{78}}\\ &= \frac{(2,3,-4)\cdot(2,-5,7)}{\sqrt{78}} = \frac{-39}{\sqrt{78}}\end{aligned}$$



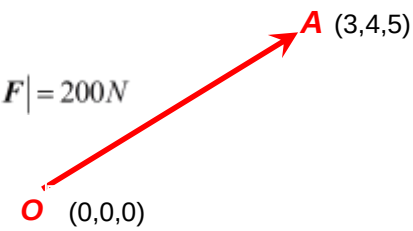
Force acting along a position vector

A force $\boldsymbol{F}$ acting along a vector whose position is $\boldsymbol{u} = (a,b,c)$ is given by

$$\boxed{\boldsymbol{F} = |\boldsymbol{F}|\hat{\boldsymbol{u}}}$$

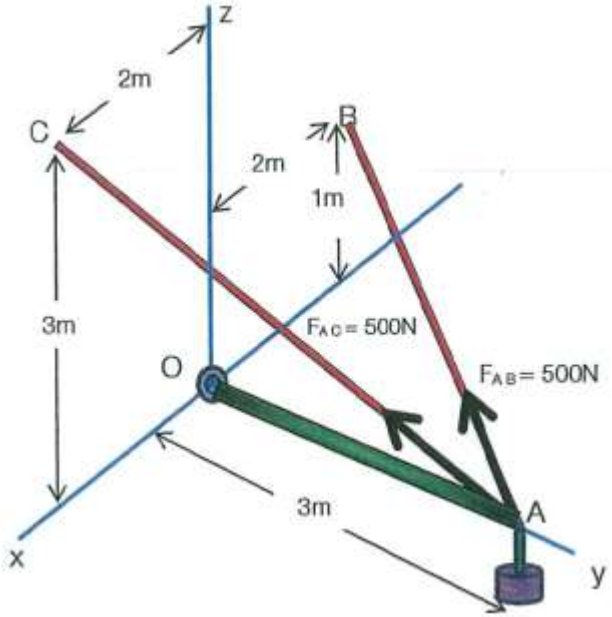
Example If *OA* represents a force $\boldsymbol{F}$ of magnitude 200N, and its position vector is $\boldsymbol{OA} = \boldsymbol{u} = (3, 4, 5)$, then vector $\boldsymbol{F}$ can be represented by

$$\begin{aligned}\boldsymbol{F} &= |\boldsymbol{F}|\hat{\boldsymbol{u}}\\ &= 200\times\frac{(3,4,5)}{\sqrt{50}}\\ &= 20\sqrt{2}(3,4,5)\\ &= \left(85\boldsymbol{i} + 113\boldsymbol{j} + 141\boldsymbol{k}\right)\mathrm{N} = (85,113,141)\,\mathrm{N}\end{aligned}$$



Example: Forces in cables

Determine (a) the vector forces in the cables $\boldsymbol{F_{AB}}$ and $\boldsymbol{F_{AC}}$ (b) the resultant vector force $\boldsymbol{F}$ of $\boldsymbol{F_{AB}}$ and $\boldsymbol{F_{AB}}$ (c) the scalar projection of $\boldsymbol{F}$ along the beam AO, (d) the perpendicular vector component of $\boldsymbol{F}$ to the beam AO, and (e) the angle between the cables AB and AC.



(a) We need to find $\boldsymbol{F_{AB}}$ and $\boldsymbol{F_{AC}}$, so we must use $\boldsymbol{F} = |\boldsymbol{F}|\hat{u}$. We know the magnitudes of each force $|\boldsymbol{F}|$ are 500N so we must find $\hat{\boldsymbol{u}}$

Position vector ***AB*** first

$\boldsymbol{OA} = (0,3,0)$; $\boldsymbol{OB} = (-2,0,1)$

$$\boldsymbol{AB} = \boldsymbol{OB} - \boldsymbol{OA}$$
$$= (-2,0,1) - (0,3,0) = (-2,-3,1)$$

Let $\hat{\boldsymbol{u}}$ be the unit vector of ***AB*** where

$$|\boldsymbol{AB}| = \sqrt{(-2)^2 + (-3)^2 + 1^2} = \sqrt{14}$$
$$\hat{\boldsymbol{u}} = \frac{\boldsymbol{AB}}{|\boldsymbol{AB}|} = \frac{1}{\sqrt{14}}(-2,-3,1)$$

Hence

$$\boldsymbol{F} = |\boldsymbol{F}|\hat{u}$$
$$= 500 \times \frac{(-2,-3,-1)}{\sqrt{14}} \approx (-267,-401,134)\,N$$

Position vector ***AC*** next, where

$\boldsymbol{OC} = (2,0,3)$

$$\boldsymbol{AC} = \boldsymbol{OC} - \boldsymbol{OA}$$
$$= (2,0,3) - (0,3,0) = (2,-3,3)$$

Let $\hat{\boldsymbol{v}}$ be the unit vector of ***AC*** where

$$|\boldsymbol{AC}| = \sqrt{(2)^2 + (-3)^2 + 3^2} = \sqrt{22}$$
$$\hat{\boldsymbol{v}} = \frac{\boldsymbol{AC}}{|\boldsymbol{AC}|} = \frac{1}{\sqrt{22}}(2,-3,3)$$

Hence

$$\boldsymbol{F} = |\boldsymbol{F}|\hat{v}$$
$$= 500 \times \frac{(2,-3,3)}{\sqrt{22}} \approx (213,-320,320)\,N$$

(b) Resultant vector force $\boldsymbol{F}$ is given by

$$\boldsymbol{F} = \boldsymbol{F}_{AB} + \boldsymbol{F}_{AC}$$
$$= (-267,-401,134) + (213,-320,320)$$
$$= (-54,-721,454)$$

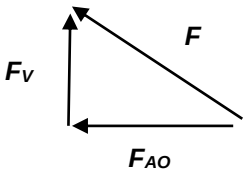
(c) Let $\boldsymbol{w}$ represent ***AO*** which is the position vector $(0,-3,0)$

Note: $\boldsymbol{OA} = (0,3,0)$ but $\boldsymbol{AO} = (0,-3,0)$

The scalar projection of ***F*** along beam *AO* is

$$|\boldsymbol{F}|_{AO} = \boldsymbol{F} \cdot \hat{\boldsymbol{w}} = \boldsymbol{F} \cdot \frac{\boldsymbol{w}}{|\boldsymbol{w}|}$$
$$= (-54,-721,454) \cdot \frac{(0,-3,0)}{\sqrt{0^2 + (-3)^2 + 0^2}}$$
$$= (-54,-721,454) \cdot (0,-1,0)$$
$$= 721N$$

(d) Perpendicular component of ***F*** to *AO* Is given by $\boldsymbol{F_V}$.



To find $\boldsymbol{F_{AO}}$

$$\boldsymbol{F}_{AO} = |\boldsymbol{F}_{AO}|\hat{u} \quad \text{where}$$

$$\hat{u} = \frac{u}{|u|} = \frac{(0,-3,0)}{\sqrt{0^2+3^2+0^2}} = (0,-1,0)$$

See part (c) above

Horizontal scalar component of ***F*** along *AO*, $\boldsymbol{F_{AO}}$, is given by the scalar projection, which from (c) is 721N.

$$\begin{aligned}\boldsymbol{F}_{AO} &= |\boldsymbol{F}_{AO}|\hat{u}\\ &= 721\times(0,-1,0)\\ &= (0,-721,0)\,\text{N} = -721\boldsymbol{j}\ \text{N}\end{aligned}$$

Since $\boldsymbol{F} = \boldsymbol{F_{AO}} + \boldsymbol{F_V}$

$$\begin{aligned}\boldsymbol{F}_V &= \boldsymbol{F} - \boldsymbol{F}_{AO}\\ &= (-54,-721,454)-(0,-721,0)\\ &= (-54,0,454)N\end{aligned}$$

(e) Angle between the cables AB and AC is the same as the angle between the position vectors $\boldsymbol{AB}$ and $\boldsymbol{AC}$.

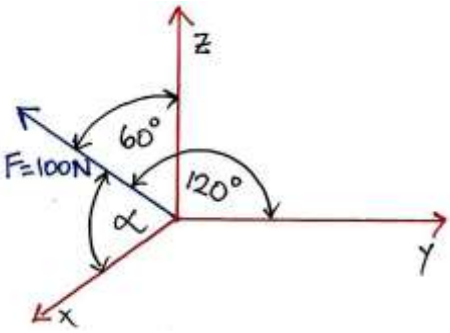
Using the scalar product

$$\boldsymbol{AB}\cdot\boldsymbol{AC} = |\boldsymbol{AB}||\boldsymbol{AC}|\cos\theta$$

$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{\boldsymbol{AB}\cdot\boldsymbol{AC}}{|\boldsymbol{AB}||\boldsymbol{AC}|}\right)\\ &= \cos^{-1}\left(\frac{(-2,-3,1)\cdot(2,-3,3)}{\sqrt{14}\sqrt{22}}\right)\\ &= \cos^{-1}\left(\frac{-4+9+3}{\sqrt{14}\sqrt{22}}\right)\\ &= \cos^{-1}\left(\frac{8}{\sqrt{308}}\right)\\ \theta &= 63^\circ\end{aligned}$$

Exercise

1(a) (a) Calculate angle α.
(b) Calculate the force vector ***F*** .

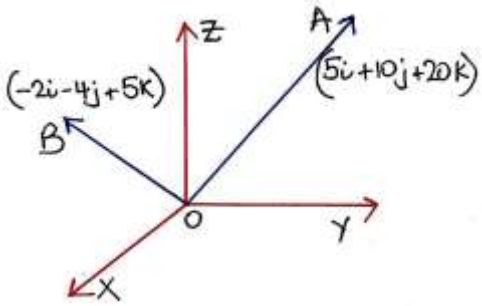


2. Refer to Q18 on the Revision of Fundamentals worksheet. Defining A as the origin of the X-Y plane, express, to 2 decimal places, ***AC*** and ***AB*** as position vectors.

3. Refer to the Cable example on page 5.
(a) Defining O as the origin, express ***OA, AB*** and ***AC*** as position vectors.
(b) Defining A as the origin, express ***AO, AB*** and ***AC*** as position vectors.

4. Express the 6 position vectors found in Question 3 as unit vectors. Give answers in exact form.

5. Position vectors ***OA*** and ***OB*** are shown at right.
(a) Find the resultant position vector of ***OA*** and ***OB***
(b) Find the unit vector acting along the resultant vector found in (a) above
(c) Find the 3 direction cosines (to the nearest degree) of the resultant vector found in (a) above.
Position vectors are in metres

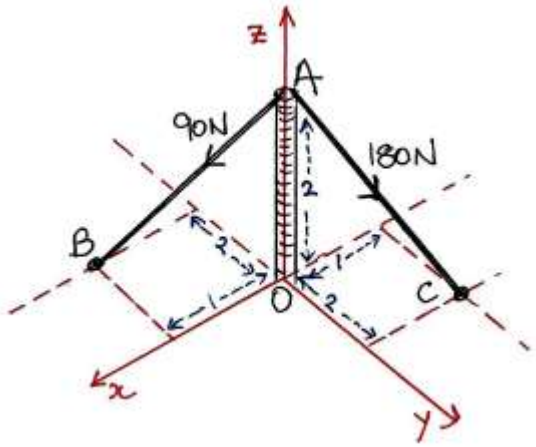


6. Refer to the diagram in Question 5.
(a) Calculate the scalar product of ***OA*** and ***OB***
(b) Calculate the angle between ***OA*** and ***OB***
(c) Calculate the scalar projection of ***OA*** on to ***OB***
(d) Calculate the scalar projection of ***OB*** on to ***OA***

7. Given $\boldsymbol{OA} = 3i - 4j + 5k$ and $\boldsymbol{OB} = 5i + 4j - 3k$ find
(a) vector ***AB***
(b) unit vector ***AB***
(c) the acute angle, to the nearest degree, between ***OA*** and ***OB***

8. Refer to the diagram in Question 5. If $\boldsymbol{F_{OA}}$ has a magnitude of 200N, and $\boldsymbol{F_{OB}}$ a magnitude 100N, calculate, to 2 decimal places
(a) Force vector $\boldsymbol{F_{OA}}$
(b) Force vector $\boldsymbol{F_{OB}}$
(c) the resultant force vector ***F*** of $\boldsymbol{F_{OA}}$ and $\boldsymbol{F_{OB}}$

9 A tent pole is supported by 2 cables as shown in the diagram at right. Calculate
(a) the position vectors ***AB*** and ***AC***
(b) the vector forces in the 2 cables $\boldsymbol{F_{AB}}$ and $\boldsymbol{F_{AC}}$
(c) the angle, to the nearest degree, between the cables AB and AC
(d) The resultant vector force ***F*** of $\boldsymbol{F_{AB}}$ and $\boldsymbol{F_{AC}}$
(e) The scalar projection of ***F*** along the tent pole AO.
(f) The horizontal component of ***F*** to the tent pole.
Dimensions are in metres.



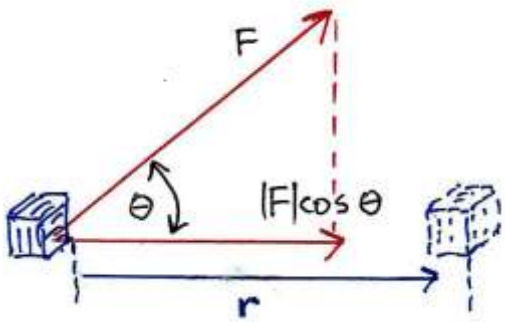
10 The work done W (in Joules) by a constant force ***F*** is the scalar product of the force ***F*** and displacement vector ***r***, ie

$W = \boldsymbol{F} \cdot \boldsymbol{r} = |\boldsymbol{F}||\boldsymbol{r}| \cos\theta$

A force of magnitude 10N inclined at an angle of 60°to the horizontal pulls an object 5m horizontally along a surface.

(a) Calculate the work done by the force using the vector representation $W = \boldsymbol{F} \cdot \boldsymbol{r}$

(b) Calculate the work done by the force using the scalar representation $W = |\boldsymbol{F}||\boldsymbol{r}| \cos\theta$



Answers

1 (a) 45°(b) $71i - 50j + 50k$

2 $\boldsymbol{AC} = (-1i + 1j)m;\ \boldsymbol{AB} = (-1i - 0.5j)m$

3. ***OA*** = (0,3,0); ***OB*** = (-2,0,1);
OC = (2,0,3);
AO = (0,-3,0); ***AB*** = (-2,-3,1);
AC = (2,-3,3);

4. Unit Vectors are:

$\boldsymbol{OA} = (0,1,0); \boldsymbol{OB} = \frac{1}{\sqrt{5}}(-2,0,1)$

$\boldsymbol{OC} = \frac{1}{\sqrt{13}}(2,0,3)$

Unit Vectors

$\boldsymbol{AO} = (0,-1,0); \boldsymbol{AB} = \frac{1}{\sqrt{14}}(-2,-3,1)$

$\boldsymbol{AC} = \frac{1}{\sqrt{22}}(2,-3,3)$

5. (a) (3,6,25) (b) $\frac{3}{\sqrt{670}}i + \frac{6}{\sqrt{670}}j + \frac{25}{\sqrt{670}}k$ (c) $\alpha = 83°$ $\beta = 77°$ $\gamma = 15°$

6. (a) 50 (b) 71° (c) 7.45
(d) 2.18

7. (a) $\boldsymbol{AB} = 2i + 8j - 8k$

(b) Unit Vector $= \frac{1}{2\sqrt{33}}(2i + 8j - 8k)$

(c) 71°

8. (a) $\boldsymbol{F}_{OA} = 43.64i + 87.29j + 174.57k$ (b) $\boldsymbol{F}_{OB} = -29.81i - 59.63j + 74.54k$
(c) $\boldsymbol{F}_R = 73.45i + 27.66j + 249.11k$

9. (a) ***AB*** = (1,-2,-2); ***AC*** = (-1,2,-2) (b) $\boldsymbol{F}_{AB}$ = (30,-60,-60)N $\boldsymbol{F}_{AC}$ = (-60,120,-120)N
(c) 96° (d) $\boldsymbol{F}_R$ = (-30,60,-180)N (e) 180N (f) (-30, 60, 0)N

10. (a) 25J (b) 25J

Additional Questions

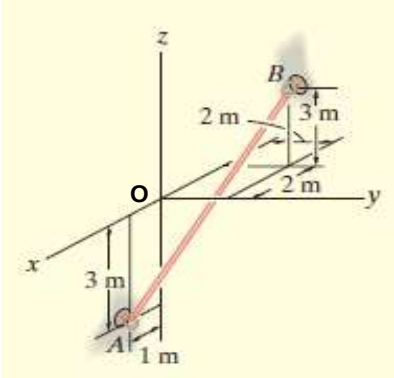
Answers to 1 decimal place

1 Refer to Q2.1, 2.1 and 2.3 on the **Revision of Fundamentals** worksheet.
Defining the screw eye or the hook as the origin of the *x-y* plane, calculate the resultant force acting on the screw eye or the hook using $\boldsymbol{i} - \boldsymbol{j}$ notation.

2 Refer to Q2.4 on the **Revision of Fundamentals** worksheet. If F_{AB} = 186N, F_{AC} = 186N, θ_1 = 30°, θ_2 = 40°, calculate $\boldsymbol{F}$ as a force vector using $\boldsymbol{i} - \boldsymbol{j}$ notation.

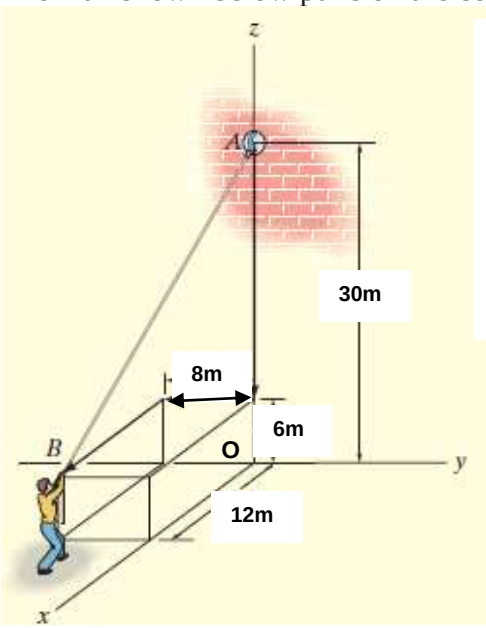
3 Refer to Q5 on the **Moments and Couples** worksheet. Calculate
(a) the direction cosines of $\boldsymbol{F_1}$
(b) the resultant force $\boldsymbol{F}$ of $\boldsymbol{F_1}$ and $\boldsymbol{F_2}$
(c) the angle, to the nearest degree, between forces $\boldsymbol{F_1}$ and $\boldsymbol{F_2}$
(d) the scalar projection of (i) $\boldsymbol{F_1}$ along AB (ii) $\boldsymbol{F_2}$ along BA.
(e) the perpendicular vector component of $\boldsymbol{F_1}$ to AB.

4 An elastic rubber band is attached to points *A* and *B* as shown below.



(a) Calculate the position vector $\boldsymbol{AB}$
(b) Calculate the length *AB*.
(c) If the magnitude of the force along AB is 98N, calculate the force vector $\boldsymbol{F_{AB}}$

5 The man shown below pulls on the cord with a force of 56kN. Calculate



(a) the position vector $\boldsymbol{AB}$

(b) the length of the cord *AB*.

(c) the force vector $\boldsymbol{F_{AB}}$

Answers

1 Q2.1. $\boldsymbol{F_R}$ = (-1.6$\boldsymbol{i}$ – 6.6$\boldsymbol{j}$)kN
Q2.2. $\boldsymbol{F_R}$ = (344.2$\boldsymbol{i}$ – 569.8$\boldsymbol{j}$)N
Q2.1. $\boldsymbol{F_R}$ = (519.6$\boldsymbol{i}$ + 500$\boldsymbol{j}$)N

2 -350$\boldsymbol{j}$ N

3 (a) cos α = -0.9; cos β = 0.2; cos γ = 0.4
(b) (-30$\boldsymbol{i}$ + 30$\boldsymbol{j}$ –10$\boldsymbol{k}$)kN
(c) θ = 128° (d) (i) 10kN (ii) -20kN

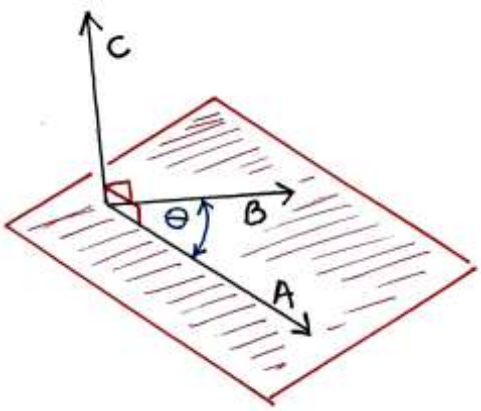
4 (a) (-3$\boldsymbol{i}$ + 2$\boldsymbol{j}$ + 6$\boldsymbol{k}$)m (b) 7m
(c) (-42$\boldsymbol{i}$ + 28$\boldsymbol{j}$ + 84$\boldsymbol{k}$)N

5 (a) (12$\boldsymbol{i}$ – 8$\boldsymbol{j}$ – 24$\boldsymbol{k}$)m (b) 28m
(b) (c) (24$\boldsymbol{i}$ – 16$\boldsymbol{j}$ – 48$\boldsymbol{k}$)kN

MOMENTS and COUPLES

Cross Product

Vectors ***A*** and ***B*** inclined at an angle θ to each other define a plane as shown in the diagram.



The cross product of ***A*** and ***B*** is vector ***C*** which will be orthogonal to the plane that is defined by ***A*** and ***B*** .

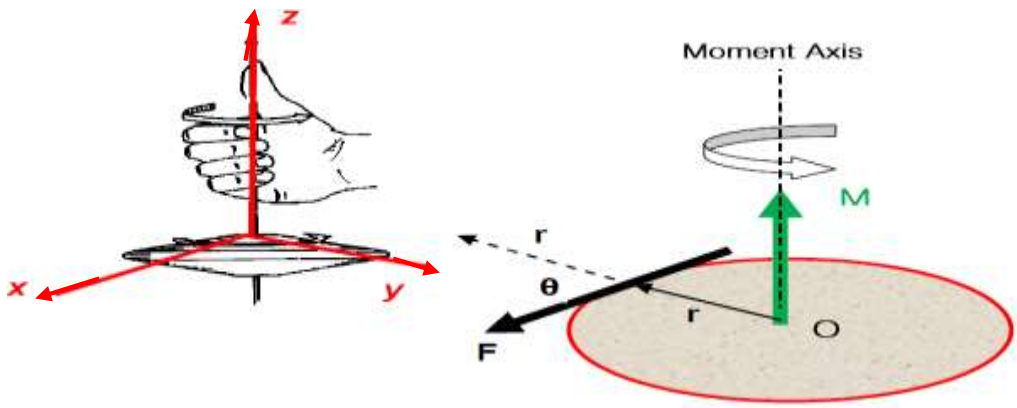
Right Hand Rule
To find the direction of vector ***C***, curl your fingers from ***A*** to ***B***, as you would if you were gripping a stick with your thumb pointing along the stick (see also diagram below). The direction of vector ***C*** is the direction that your thumb would point - in this case, upwards. It can be written in vector form as

$$\boxed{\boldsymbol{C} = \boldsymbol{A} \times \boldsymbol{B}}$$

which is read "***A*** cross ***B***"

Note that order here is important: cross product is non-commutative, ie $\boldsymbol{A} \times \boldsymbol{B} \neq \boldsymbol{B} \times \boldsymbol{A}$.
Using the right hand rule for ***B*** cross ***A*** results in vector ***C*** that will point downwards, or

$$-\boldsymbol{C} = \boldsymbol{B} \times \boldsymbol{A}$$



The vector cross product can also be defined in scalar from as

$$\boxed{\boldsymbol{C} = |\boldsymbol{A}||\boldsymbol{B}| \sin\theta}$$

where $|\boldsymbol{A}|$ and $|\boldsymbol{B}|$ are the magnitudes of vectors ***A*** and ***B***, and $\sin\theta$ is the angle between them.

Note that if ***A*** and ***B*** are parallel then the cross product is zero. For unit vectors, this becomes

$$\boldsymbol{i} \times \boldsymbol{i} = \boldsymbol{j} \times \boldsymbol{j} = \boldsymbol{k} \times \boldsymbol{k} = 0$$

If ***A*** and ***B*** are at right angles to each other then the cross product will be a vector orthogonal to ***A*** and ***B***.
For unit vectors, this can be written

$$\boldsymbol{i} = \boldsymbol{j} \times \boldsymbol{k} \qquad \boldsymbol{j} = \boldsymbol{k} \times \boldsymbol{i} \qquad \boldsymbol{k} = \boldsymbol{i} \times \boldsymbol{j}$$

using the right hand rule
Evaluating cross products is normally carried out using determinants:

$$\boldsymbol{A}\times\boldsymbol{B}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \boldsymbol{A}_x & \boldsymbol{A}_y & \boldsymbol{A}_z\\ \boldsymbol{B}_x & \boldsymbol{B}_y & \boldsymbol{B}_z\end{vmatrix}$$

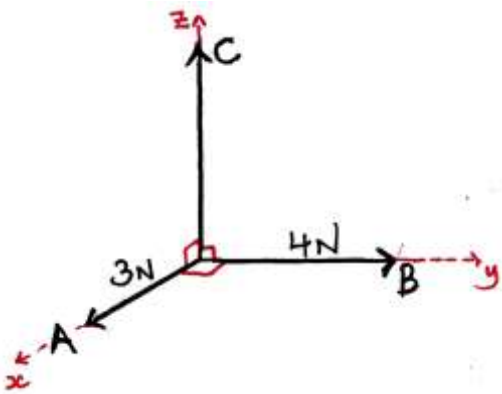
where $\boldsymbol{A}_x$ **,** $\boldsymbol{A}_y$, $\boldsymbol{A}_z$ are the *x*, *y* and *z* components of vectors ***A*** respectively**.** The solution to this determinant is

$$\boldsymbol{A}\times\boldsymbol{B}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \boldsymbol{A}_x & \boldsymbol{A}_y & \boldsymbol{A}_z\\ \boldsymbol{B}_x & \boldsymbol{B}_y & \boldsymbol{B}_z\end{vmatrix}=\boldsymbol{i}\left(\boldsymbol{A}_y\boldsymbol{B}_z-\boldsymbol{A}_z\boldsymbol{B}_y\right)-\boldsymbol{j}\left(\boldsymbol{A}_x\boldsymbol{B}_z-\boldsymbol{A}_z\boldsymbol{B}_x\right)+\boldsymbol{k}\left(\boldsymbol{A}_x\boldsymbol{B}_y-\boldsymbol{A}_y\boldsymbol{B}_x\right)$$

Example

Two vectors ***A*** and ***B*** are acting as shown in the diagram.
Expressing these vectors in component form:

$\boldsymbol{A}=(3\boldsymbol{i}+0\boldsymbol{j}+0\boldsymbol{k})\ \mathrm{N}$ $\qquad$ $\boldsymbol{B}=(0\boldsymbol{i}+4\boldsymbol{j}+0\boldsymbol{k})\ \mathrm{N}$



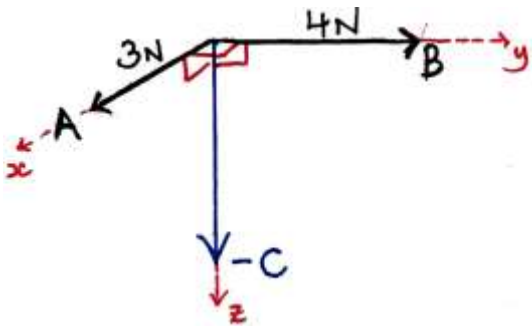
$$\boldsymbol{A}\times\boldsymbol{B}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ 3 & 0 & 0\\ 0 & 4 & 0\end{vmatrix}$$

$$=\boldsymbol{i}\left(0\times0-0\times4\right)-\boldsymbol{j}\left(3\times0-0\times0\right)+\boldsymbol{k}\left(3\times4-0\times0\right)$$

$$=(0\boldsymbol{i}-0\boldsymbol{j}+12\boldsymbol{k})=12\boldsymbol{k}$$

The cross product is 12***k***, ie vector C is 12***k*** in the positive ***z***-direction.

Expressing the vector product $\boldsymbol{B}\times\boldsymbol{A}$ would lead to the following



$$\boldsymbol{B}\times\boldsymbol{A}=\begin{vmatrix}i & j & k\\ 0 & 4 & 0\\ 3 & 0 & 0\end{vmatrix}$$

$$=\boldsymbol{i}\left(4\times0-0\times0\right)-\boldsymbol{j}\left(0\times0-0\times3\right)+\boldsymbol{k}\left(0\times0-4\times3\right)$$

$$=(0\boldsymbol{i}-0\boldsymbol{j}-12\boldsymbol{k})=-12\boldsymbol{k}$$

The cross product is -12***k***, ie 12 units in the negative ***z***-direction, or $-C$ as shown in the diagram. Note again that order here is important.
Using the right hand rule for ***A*** cross ***B*** results in vector 12***k*** which points upwards, whereas ***B*** cross ***A*** results in $-12\boldsymbol{k}$ which points downwards.

Moment of force (or torque)

The moment $\boldsymbol{M}$ (or torque τ) of a force $\boldsymbol{F}$ acting at a displacement $\boldsymbol{r}$ from the axis of rotation O (or pivot point) of a rigid body is given by

$$\boxed{\boldsymbol{M}_o = \boldsymbol{r} \times \boldsymbol{F} = |\boldsymbol{r}||\boldsymbol{F}|\sin\theta}$$

where θ is the angle between $\boldsymbol{F}$ and $\boldsymbol{r}$. $\boldsymbol{r}$ is called the moment arm.

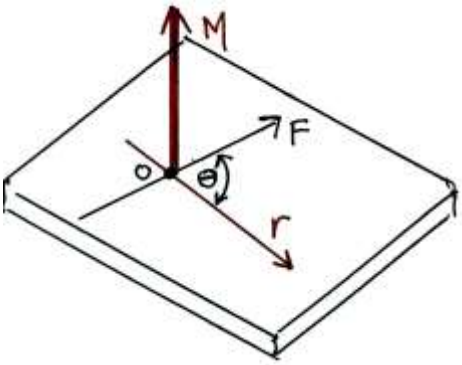
A maximum moment is achieved when θ = 90°. So $\boldsymbol{M}_O = |\boldsymbol{r}||\boldsymbol{F}|\sin\theta$ becomes

$$\boxed{\boldsymbol{M}_O = |\boldsymbol{r}|_\perp |\boldsymbol{F}|}$$

where the moment arm is perpendicular to the force for maximum moment. See diagram below right.

Maximum Moment

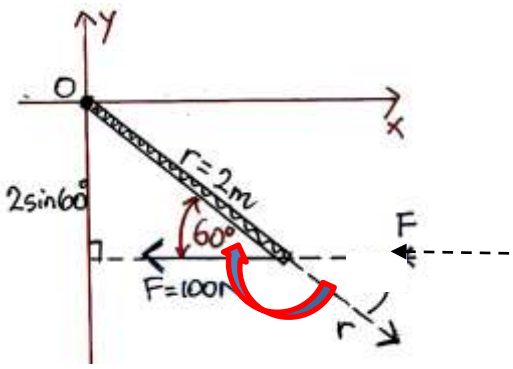
For example, opening a door requires a force $\boldsymbol{F}$ acting on the door handle a distance r from the hinge or pivot point. A maximum moment will be achieved if this force acts at right angles to the door. Applying the same force $\boldsymbol{F}$ closer to the hinge (shorter moment arm) and/or applying a force which is not perpendicular to the door will make it harder (larger moment required) for you to open the door.



The direction of the moment is found by using the right hand rule. In the diagram at right it can be seen that the direction is upwards.

Example 2-dimensional representation

Refer to diagram at right. Calculate the moment of the force $\boldsymbol{F}$ about O, where $\boldsymbol{F}$ = 100N is the force parallel to the *x*-axis.



(a) Using a scalar analysis

$\boldsymbol{M}_O = |\boldsymbol{r}|_\perp |\boldsymbol{F}|$ where $|\boldsymbol{r}|_\perp$ and $|\boldsymbol{F}|$ are at right angles to each other. Here $|\boldsymbol{r}|_\perp$ is the *y*-component of $\boldsymbol{r}$ or $2\sin 60°$. Using trigonometry

$$\boldsymbol{M}_O = 100 \times 2\sin 60° = 173\text{Nm}.$$

The direction is found using the right hand rule which shows the ***tip*** of position vector $\boldsymbol{r}$ rotating to the ***tip*** of force vector $\boldsymbol{F}$ in a clockwise direction. Hence, the moment is into the page.

(b) Using a vector analysis

$$\boldsymbol{M}_o = \boldsymbol{r} \times \boldsymbol{F}$$

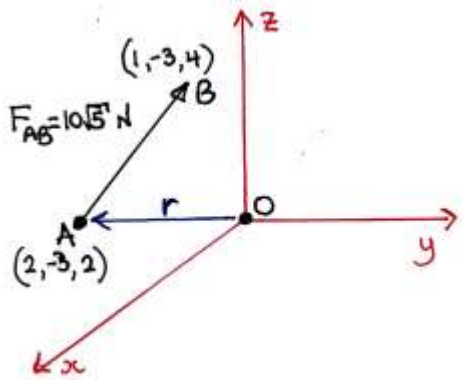
where $\boldsymbol{r} = 2\cos 60\boldsymbol{i} - 2\sin 60\boldsymbol{j} + 0\boldsymbol{k} = (1, -\sqrt{3}, 0)$ and $\boldsymbol{F} = -100i + 0j + 0k = (-100, 0, 0)$. Hence

$$\boldsymbol{M}_o = \boldsymbol{r} \times \boldsymbol{F} = (1, -\sqrt{3}, 0) \times (100, 0, 0) = -173k \text{ Newton metres}$$

Check this answer using determinants. Note that in problems in 2-dimensions the $\boldsymbol{k}$ component is zero, and the moment of -173$\boldsymbol{k}$ indicates the direction of $\boldsymbol{M}_o$ is into the page as shown in the scalar analysis.

Example 3-dimensional representation

A force $\boldsymbol{F_{AB}}$ of magnitude $10\sqrt{5}$ N is acting as shown in the diagram. Find the moment of $\boldsymbol{F_{AB}}$ about O.



Since $\boldsymbol{M}_0 = \boldsymbol{r} \times \boldsymbol{F_{AB}}$ we need to find $\boldsymbol{r}$ and $\boldsymbol{F_{AB}}$

Position vector first

$$\boldsymbol{r} = \boldsymbol{OA} = (2, -3, 2) - (0, 0, 0) = (2, -3, 2)$$

To find the vector $\boldsymbol{F}_{AB}$ we need to use $\boldsymbol{F}_{AB} = |\boldsymbol{F}_{AB}|\hat{\boldsymbol{u}}$ where $\hat{\boldsymbol{u}}$ is the unit vector ***AB***

We know $|\boldsymbol{F}_{AB}| = 10\sqrt{5}$ N, so we need to find $\hat{\boldsymbol{u}}$

$$\boldsymbol{AB} = \boldsymbol{OB} - \boldsymbol{OA}$$
$$= (1, -3, 4) - (2, -3, 2) = (-1, 0, 2)$$

$$|\boldsymbol{AB}| = \sqrt{(-1)^2 + (0)^2 + 2^2} = \sqrt{5}.$$

Hence $\hat{\boldsymbol{u}} = \dfrac{\boldsymbol{AB}}{|\boldsymbol{AB}|} = \dfrac{1}{\sqrt{5}}(-1, 0, 2)$.

Since $\boldsymbol{F}_{AB} = |\boldsymbol{F}_{AB}|\hat{\boldsymbol{u}}$, $\quad \boldsymbol{F}_{AB} = 10\sqrt{5} \times \dfrac{(-1, 0, 2)}{\sqrt{5}} = (-10, 0, 20)$ or $(-10i + 20k)$ N

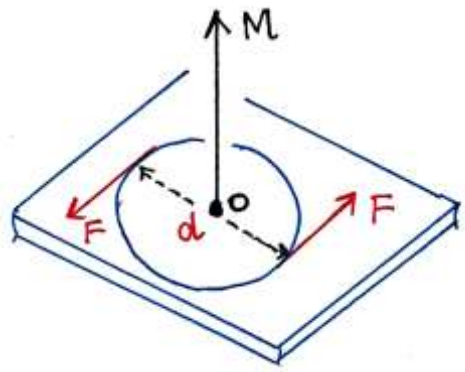
Now $\boldsymbol{M}_0 = \boldsymbol{r} \times \boldsymbol{F_{AB}}$, and evaluating using the determinant we get

$$\boldsymbol{M}_0 = \boldsymbol{r} \times \boldsymbol{F_{AB}} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & -3 & 2 \\ -10 & 0 & 20 \end{vmatrix} = -60\boldsymbol{i} - 60\boldsymbol{j} - 30\boldsymbol{k}$$

$$\boldsymbol{M}_0 = (-60\boldsymbol{i} - 60\boldsymbol{j} - 30\boldsymbol{k}) \text{ Newton metres}$$

Moment of a couple

A couple is a special case of a moment. It can be defined as two equal and opposite forces that do not act through the same point. Consequently, a turning effect is achieved, even though there is no resultant force. Referring to the diagram, taking moments about O, and letting counterclockwise moments be positive.



Scalar Approach

$$M = F \times \frac{d}{2} + F \times \frac{d}{2} = F\,d$$

where d is the diameter and F the force.

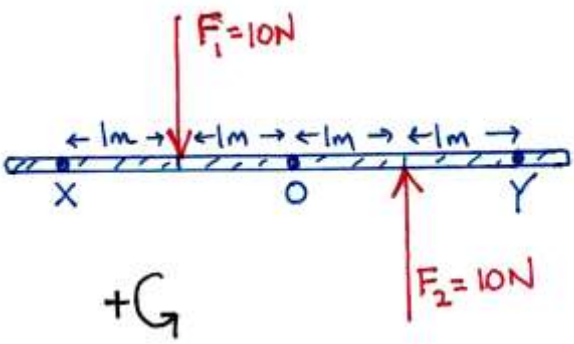
Hence The moment of a couple is given by

$$\boxed{M_0 = F_\perp\, d}$$

Note the direction of the couple (upwards), as derived from the right hand rule.

The moment of a couple depends only on the distance between the forces. Regardless of where moments are taken about, the couple moment still remains the same.

Example Consider the beam in the diagram, with two equal but opposite forces, and taking moments about three different positions along the beam. Take counterclockwise moments as positive.



Moments about O

$M_o = (F_2 \times 1) + (F_1 \times 1) = (10 \times 1) + (10 \times 1) = +20$ Nm

Moments about X

$M_x = (F_2 \times 3) + (-F_1 \times 1) = (10 \times 3) + (-10 \times 1) = +20$ Nm

Moments about Y

$M_x = (-F_2 \times 1) + (F_1 \times 3) = (-10 \times 1) + (10 \times 3) = +20$ Nm

Vector Approach

The description above is from a scalar perspective. Using a vector approach, the moment of a couple is given by the sum of the vector cross products, or

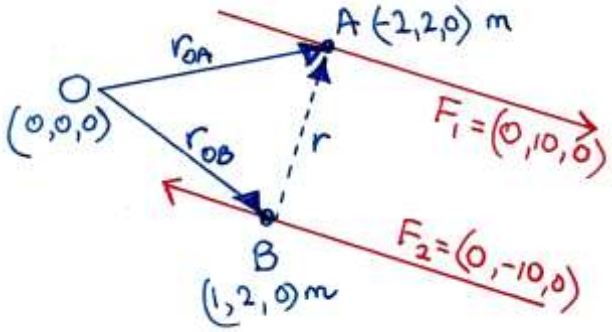
$$\boldsymbol{M}_o = (\boldsymbol{r}_1 \times \boldsymbol{F}_1) + (\boldsymbol{r}_2 \times \boldsymbol{F}_2)$$

where $\boldsymbol{F}_1 = -\boldsymbol{F}_2$ are the opposing forces, and $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are the position vectors from any common point to the forces. This can be summarised as

$$\boxed{\boldsymbol{M}_o = \boldsymbol{r} \times \boldsymbol{F}}$$

where $\boldsymbol{r}$ is the distance between the opposing forces $\boldsymbol{F}$

Example Calculate the couple moment acting at point O in the diagram below.



$$\boldsymbol{M}_O = (\boldsymbol{r}_{OA} \times \boldsymbol{F}_1) + (\boldsymbol{r}_{OB} \times \boldsymbol{F}_2)$$

$$\boldsymbol{M}_O = (-2,2,0)\times(0,10,0) + (1,2,0)\times(0,-10,0) = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -2 & 2 & 0 \\ 0 & 10 & 0 \end{vmatrix} + \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 2 & 0 \\ 0 & -10 & 0 \end{vmatrix}$$

$$= -30\boldsymbol{k} \text{ Nm}$$

Note: The direction of the couple moment is into the page. Check this using the right hand rule.

Alternatively: take couple moments about A and B. Your answers should be the same, ie -30***k*** Nm

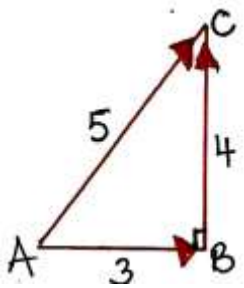
Exercise

1. Calculate the cross product of $\boldsymbol{AB} \times \boldsymbol{BC}$, $\boldsymbol{BC} \times \boldsymbol{AC}$ and $\boldsymbol{AB} \times \boldsymbol{AC}$ using a

(a) Scalar analysis

(b) Vector analysis

Magnitude and Direction required for each

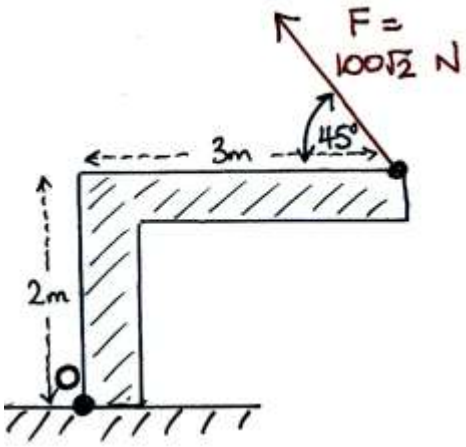


2. Calculate the moment of the force ***F*** about O

(a) Using a scalar analysis

(b) Using a vector analysis

Magnitude and Direction required for each

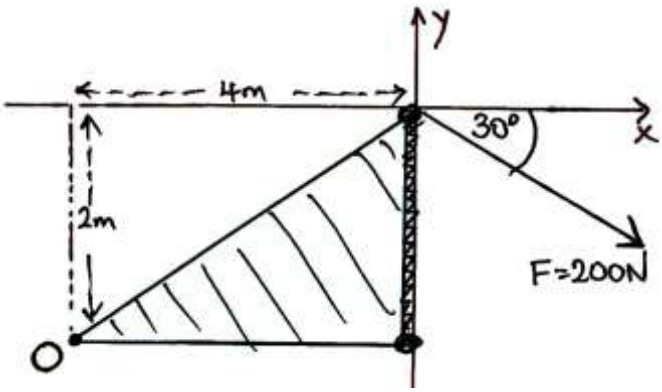


3. A triangular flag is being supported by a rope with a force F = 200N. Calculate the moment of the force about O

(a) Using a scalar analysis (magnitude and direction)

(b) Using a vector analysis (magnitude and direction)

Answers to 1 decimal place.



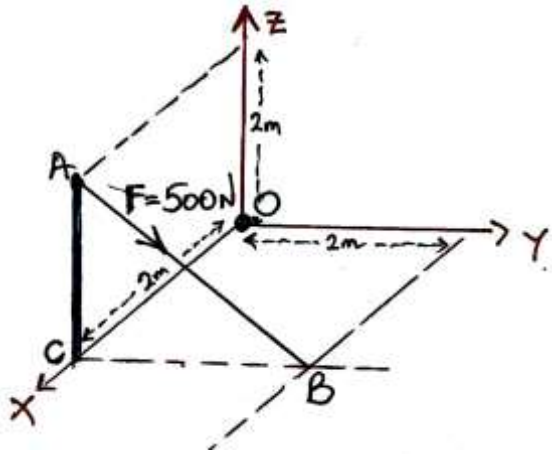
4. A cable AB is holding up a post AC with a force of ***F*** = 500N. Calculate

(a) Position vectors $\boldsymbol{r}_{OA}, \boldsymbol{r}_{OB}, \boldsymbol{r}_{OC}, \boldsymbol{r}_{AC}, \boldsymbol{r}_{AB}$

(b) The Cartesian vector ***F***

Recall $\boldsymbol{F} = |\boldsymbol{F}|\hat{\boldsymbol{u}}$

(c) the moment of ***F*** about O.

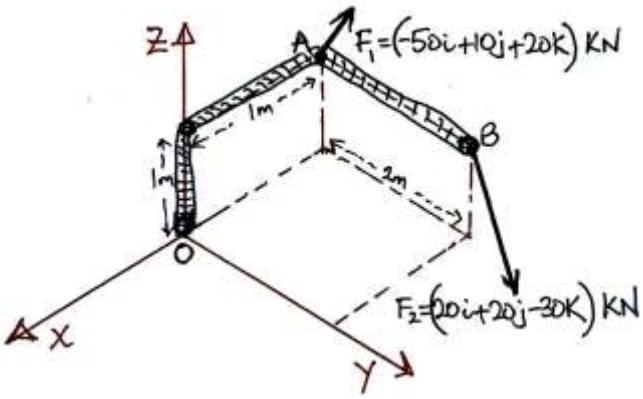


5. A tubular pipe structure OAB is shown in the diagram at right. It is subjected to two forces $\boldsymbol{F_1}$ = (-50***i*** +10***j*** +20***k***)kN and $\boldsymbol{F_2}$ = (20***i*** +20***j*** -30***k***)kN. Calculate

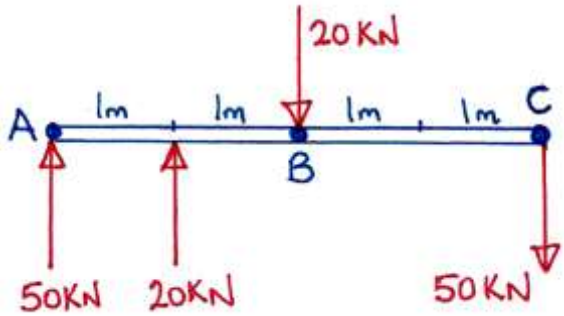
(a) The moment $\boldsymbol{M_1}$ produced on the structure by $\boldsymbol{F_1}$ about point O.

(b) The moment $\boldsymbol{M_2}$ produced on the structure by $\boldsymbol{F_2}$ about point O.

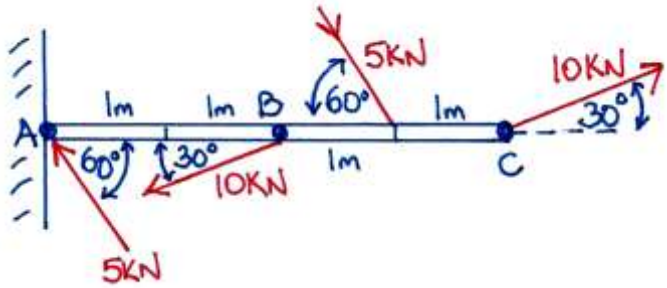
(c) The resultant moment ***M*** produced by $\boldsymbol{F_1}$ and $\boldsymbol{F_2}$ about O.



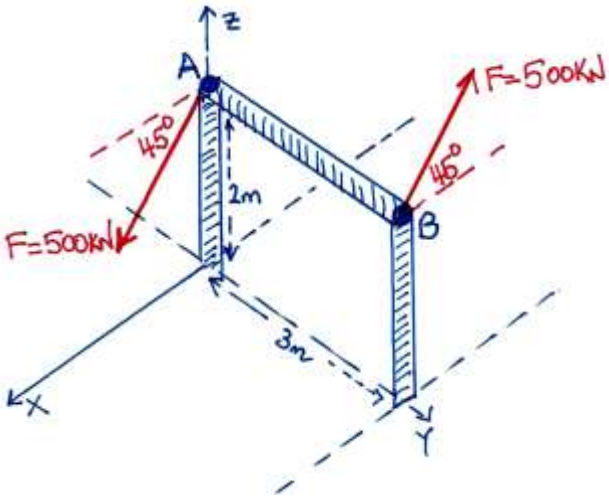
6. Calculate the resultant couple moment acting on the beam about the three points A, B and C, and show that they are equal.



7. Calculate the resultant couple moment acting on the beam about points A, B and C, and show that they are equal.



8. Calculate the couple moment acting on the structure at A and B, and show that they are equal.



Answers

1 $\boldsymbol{AB}\times\boldsymbol{BC}=12$ out of page. $\boldsymbol{BC}\times\boldsymbol{AC}=12$ into page. $\boldsymbol{AB}\times\boldsymbol{AC}=12$ out of page

2. 500Nm out of page 3. 746.4N into page

4. (a) $\boldsymbol{r}_{OA}=(2,0,2)\text{m},\ \boldsymbol{r}_{OB}=(2,2,0)\text{m},\boldsymbol{r}_{OC}=2,0,0)\text{m},\boldsymbol{r}_{AC}=(0,0,-2)\text{m},\boldsymbol{r}_{AB}=(0,2,-2)\text{m}$
(b) (0,353.5,-353.5) N (c) (-707.1***i*** + 707.1***j*** + 707.1***k***)Nm

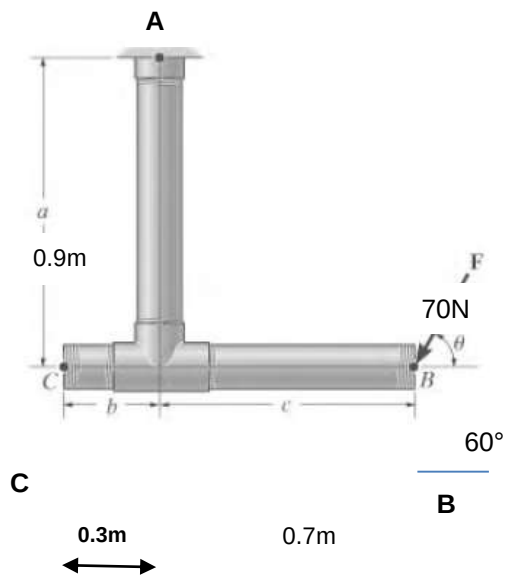
5. (a) $\boldsymbol{M_1}$ = (-10, -30, -10) kNm (b) $\boldsymbol{M_2}$ = (-80, -10, -60) kNm (c) $\boldsymbol{M}$ = (-90, -40, -70) kNm

6. 29Nm 7. (-10***i***+13***j***) Nm 8. (1500i + 1000j -1500k)kNm

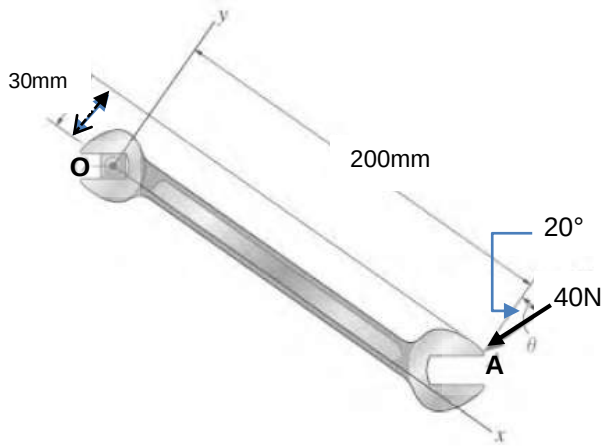
Additional Questions

Answers to 1 decimal place

1 A force of 70N acts on the end of the pipe at *B*. Determine (a) the moment of this force about point *A* using both scalar and vector analysis, and (b) the magnitude and direction of a horizontal force, applied at *C*, which stops the rotation of the pipe caused by the 70N force at B.



2 A force of 40N is applied to the wrench at A. Determine the moment of this force about point *O*. Solve the problem using both a scalar analysis and a vector analysis.



3 Refer to Q 9 (tent pole) in the **Moments and Couples** worksheet. Calculate

(a) the moment $\boldsymbol{M_{AB}}$ produced on the tent pole by $\boldsymbol{F_{AB}}$ about point O.

(b) the moment $\boldsymbol{M_{AC}}$ produced on the tent pole by $\boldsymbol{F_{AC}}$ about point O.

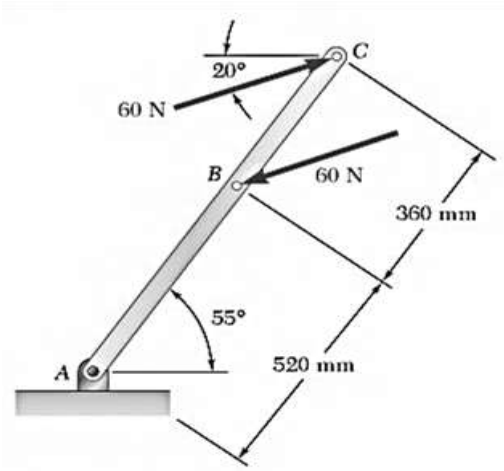
(c) the moment $\boldsymbol{M_R}$ produced on the tent pole by the resultant force $\boldsymbol{F_R}$ about point O.

4 Two parallel 60N forces are applied to a lever as shown. Calculate the couple moment of the forces by

(a) resolving each force into horizontal and vertical components and taking moments about a convenient point.

(b) using the perpendicular distance between the two forces.

(c) summing the moments of the two forces about A using a vector analysis.



Answers

1 (a) 73.9Nm (b) 82.1N into the pipe.

2 7.1Nm

3 (a) (120***i*** +60***j***)Nm (b) (-240***i*** – 120***j***)Nm (c) (-120***i*** – 60***j***)Nm

4 (a) 12.4Nm (b) 12.4Nm (c) 12.4Nm

MOTION: CONSTANT and NON-CONSTANT ACCELERATION

Constant acceleration: Graphing motion

Displacement s (m) Vector

***s* = Final position − Initial position**

Displacement is also referred to as ***position***

Distance d (m) Scalar

Distance is the magnitude of the displacement. It has no direction.

Velocity v (m/s) Vector

Velocity is the time rate of change of displacement.

$v = \frac{\Delta s}{\Delta t}$ where Δs is the change in displacement, and Δt is the change in time.

Average Velocity *(m/s)*

$v = \frac{\Delta s}{\Delta t}$ **Note**: Velocity, or *instantaneous velocity*, is calculated at a *particular time*, whereas average velocity is calculated over a given *time interval*.

Speed $|\boldsymbol{v}|$ **(m/s)** ***Scalar***

Speed is the magnitude of velocity: $|v| = \frac{\Delta d}{\Delta t}$, where Δd is the change in distance, and Δt is the change in time.

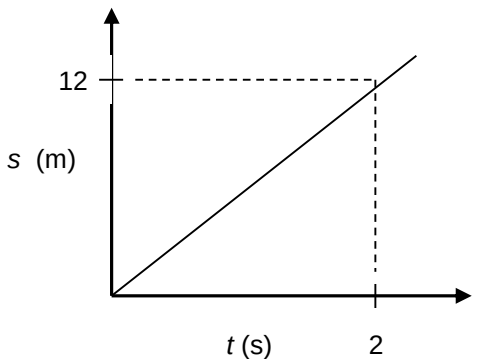
Average Speed $|\boldsymbol{v}_{AV}|$ (m/s) Scalar

$$|\boldsymbol{v}_{AV}| = \frac{\text{Total Distance}}{\text{Total Time}} = \frac{d_{Total}}{t_{Total}}$$

Acceleration a (m/s^2) Vector

Acceleration is the time rate of change of velocity, ie $a = \frac{\Delta v}{\Delta t}$.

Displacement-Time graphs



A person runs 12 metres in an easterly direction in 2 seconds.

$$v = \frac{\Delta s}{\Delta t} = \frac{12}{2} = +6\,ms^{-1}$$

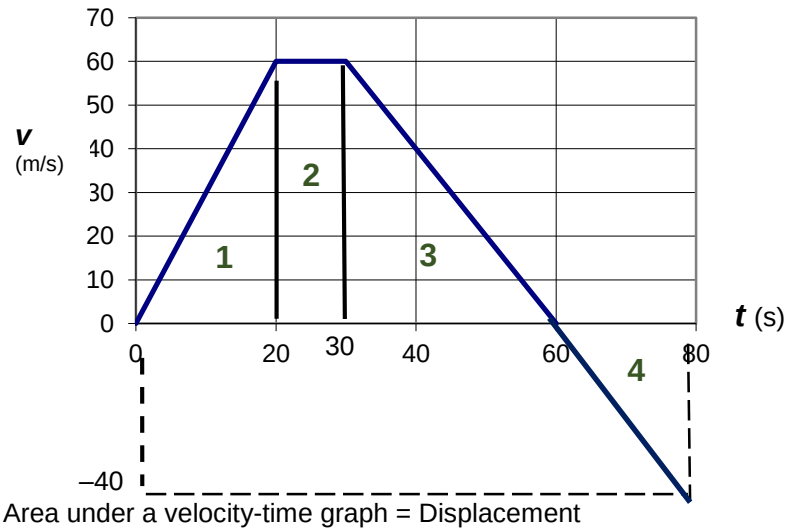
Note: the gradient of a Displacement-Time graph gives the velocity. The positive sign indicates the person is running in an easterly direction.

Gradient of displacement-time graph = velocity

Gradient of distance-time graph = speed

Velocity-Time graphs

The velocity-time graph below shows a car which accelerates uniformly from rest to 60 ms^{-1} in 20 seconds, then travels at a constant velocity of 60 ms^{-1} for the next 10 seconds, then decelerates uniformly to rest in 30 seconds. It then accelerates from rest to -40 ms^{-1} in 20s. The total journey took 80 seconds.



Area under a velocity-time graph = Displacement
Area = length × width = metres/second × seconds = metres.

Area under a velocity-time graph gives the displacement **Area under a speed-time graph gives the distance**

We can divide the graph into 4 distinct sections and calculate the area of each.

Area **1** Triangle = ½ × base × height = ½×20 × 60 = 600 metres.
Area **2** Rectangle = Length × Width = 60 × 10 = 600 metres.
Area **3** Triangle = ½ × base × height = ½×30 × 60 = 900 metres.
Area **4** Triangle = ½ × base × height = ½×20 × -40 = -400 metres.

Displacement = Area (plus and minus) = +600 + 600 + 900 – 400 = +1700 metres
Distance = Area (add everything even if negative) = 600 + 600 + 900 + 400 = 2500 metres

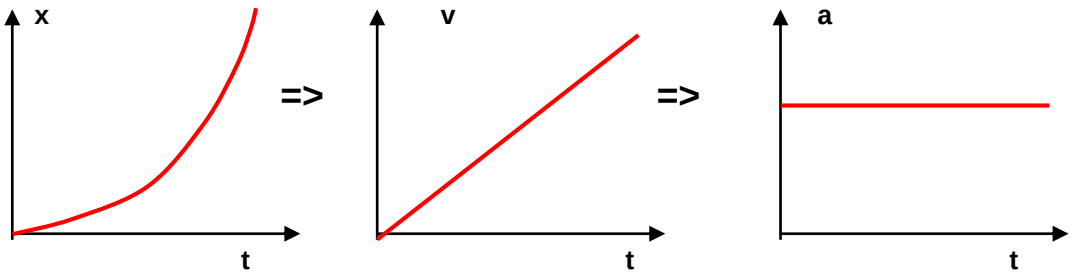
Gradient of a velocity-time graph = Acceleration

$$\textbf{Gradient} = \frac{\textbf{Rise}}{\textbf{Run}} = \frac{\textbf{Change in Velocity}}{\textbf{Time taken}} = \textbf{Acceleration}$$

From the graph above
Area **1** 0-20 seconds: Gradient = 60/20 = 3 ms^{-2}. Car is accelerating at 3 ms^{-1}.
Area **2** 20-30 seconds: Gradient = 0/10 = 0 ms^{-2}. *Note:* No slope ∴ No acceleration.
Car is travelling at a constant velocity of 60m/s.
Area **3** 30-60 seconds. Gradient = –60/30 = –2 ms^{-2} Car is decelerating at 2 ms^{-2}.
Area **4** 60-80 seconds. Gradient = –40/20 = –2 ms^{-2}. Car is accelerating at 2 ms^{-2} in the opposite direction.

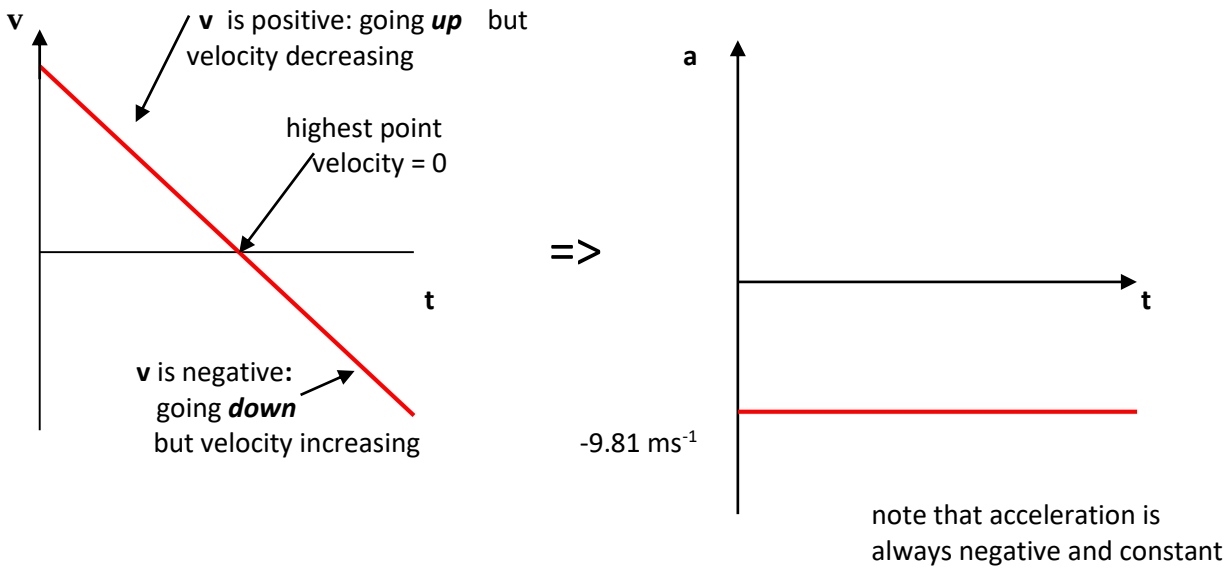
Note: Negative slope above *t*-axis: velocity is positive but car is slowing down.
Negative slope below *t*-axis: velocity is negative but car is speeding up in the opposite direction. ☹

Example: A car accelerates from a stationary position. The acceleration is constant



Note that the slope of the *s-t* graph gives a *v-t* graph, and the slope of the *v-t* graph gives an *a-t* graph. The slope of the *s-t* graph (velocity) above is found by taking the ***tangents to the curve at various points*** on the *s-t* graph.

Example A ball is thrown vertically upwards.



Summary of graphing motion

Graph type found from	***s - t***	***v – t***	***a - t***
Gradient	**Velocity**	**Acceleration**	
Area under graph		**Displacement**	**Change in velocity**

Constant acceleration: Equations of Motion

We have already used the formula $v = \frac{\Delta s}{\Delta t}$ to describe motion where the velocity is constant. This formula cannot be used in situations where there is uniform accelerated motion.

Recall from graphs of motion that the constant acceleration of an object is given by the gradient of a velocity-time graph:

$$\text{acceleration} \quad a = \frac{\text{rise}}{\text{run}} = \frac{\text{change in velocity}}{\text{time taken}} = \frac{\Delta v}{\Delta t} = \frac{v - v_0}{t} \quad \therefore \; at = v - v_0 \quad \text{or}$$

$$\boldsymbol{v = v_o + at}$$

Also, it can be shown that

$$\boldsymbol{s = v_o t + \frac{1}{2}at^2} \qquad \boldsymbol{s = vt - \frac{1}{2}at^2} \qquad \boldsymbol{v^2 = v_0^{\,2} + 2as} \qquad \boldsymbol{s = \frac{v_0 + v}{2}t}$$

where s = displacement(m), v_o = initial velocity(ms^{-1}), v = final velocity (ms^{-1}), t = time taken (s)

Note: It is often necessary to specify right or left as the positive or negative direction when doing these problems since we are dealing with vector quantities.

Example
A truck has an acceleration of $0.8ms^{-2}$. Find (a) the time taken and (b) the distance travelled for the truck to reach a velocity of $20ms^{-1}$ from a starting velocity of $3ms^{-1}$.

(a) List the data first: $v_o = 3$, $v = 20$, $a = 0.8$, $t = ?$

Now find the appropriate equation that includes this data: $v = v_o + at$

From which $t = \frac{v - v_O}{a} = \frac{20-3}{0.8} = 21.3s$

(b) List the data: $v_o = 3$, $v = 20$, $a = 0.8$, $s = ?$

Now find the appropriate equation that includes this data: $v^2 = v_o^2 + 2as$

From which $s = \frac{v^2 - v_O^2}{2a} = \frac{20^2 - 3^2}{2 \times 0.8} = 244.4m$

Vertical motion under gravity
The acceleration of a falling object near the Earth's surface is $9.8ms^{-2}$. For instance, a coin that is dropped from rest will have a velocity of $9.8ms^{-1}$ after 1 second, $19.6ms^{-1}$ after 2 seconds, and so on. The quantity, 'g', is used when describing gravitational acceleration. For example, an astronaut may experience an acceleration of $39.2ms^{-2}$, or "4g", at take-off.

Since the acceleration of a freely falling object is uniform, we can use the same equations as we did with horizontal motion.

***Note**:* As before, it is necessary to specify up or down as the positive or negative direction when doing these problems as we are dealing with vector quantities.

Example
A construction worker accidentally knocks a brick from a building so that it falls vertically a distance of 50m to the ground. Calculate (a) the time taken for the brick to reach the ground, (b) the speed of the brick as it hits the ground.

Take **up** as positive, and **down** as negative

(a) List the data first: $v_o = 0$ (at rest), $s = -50$ (from the building to the ground is a negative displacement), $a = -9.8$ (acceleration is always down, therefore negative), $t = ?$

Now find the appropriate equation that includes this data: $s = v_o t + ½ a t^2$

$s = v_o t + 0.5 \times -9.8 \times t^2 \Rightarrow -50 = 0 + 0.5 \times -9.8 \times t^2 \Rightarrow t = 3.2s$

(b) List the data first: $v_o = 0$ (at rest), $s = -50$, $a = -9.8$, $t = 3.2$, $v = ?$

Now find the appropriate equation that includes this data: $v = v_0 + at$

$v = v_o + at = 0 + -9.8 \times 3.2 \Rightarrow v = -31ms^{-1}$ The negative sign indicates that the velocity is downwards.

Example
A ball is thrown up into the air with a velocity of $30ms^{-1}$. Find (a) the maximum height reached by the ball, (b) the time taken for the ball to reach its maximum height, (c) total time of flight

Take **up** as positive, and **down** as negative.

(a) List the data: $v_o = +30$ (velocity is **up**), $v = 0$ (ball is stationary at top of flight), $a = -9.8$ (acceleration is always **down**), $s = ?$

Now find the appropriate equation that includes this data: $v^2 = v_o^2 + 2as$

$0 = 30^2 + 2 \times -9.8 \times s \Rightarrow s = 46m$

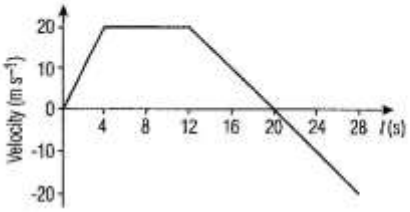
(b) List the data: $v_o = +30$ (velocity is up), $v = 0$ (ball is stationary at top of flight),

a = –9.8 (acceleration is always down), s = 45, t = ? Now find the appropriate equation that includes this data: $v = v_o + at$ $0 = 30 + -9.8 \times t$ ⇨ t = 3.1s

(c) By symmetry, the ball will also take 3.1 seconds to return to the ground from its maximum height, assuming that air resistance is ignored. Total time of flight = 6.2 seconds.

Exercise *Use g = 9.8ms^{-2}* where necessary

1 The following velocity–time graph was derived from a test drive of a prototype sports car. The car started from rest and initially travelled north.

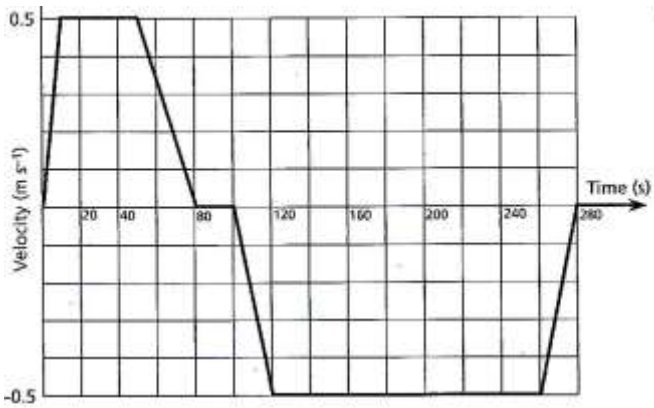


a What distance did the car travel (i) during the first four seconds of its motion, (ii) between 4 s and 12 s, and (iii) between 12 s and 28 s?
b Calculate the displacement of the car after 28 s.
c What was the total distance that the car travelled after 28 s?
d What was the average speed of the car during the 28 s interval?
e Calculate the acceleration of the car between 4 s and 12 s.

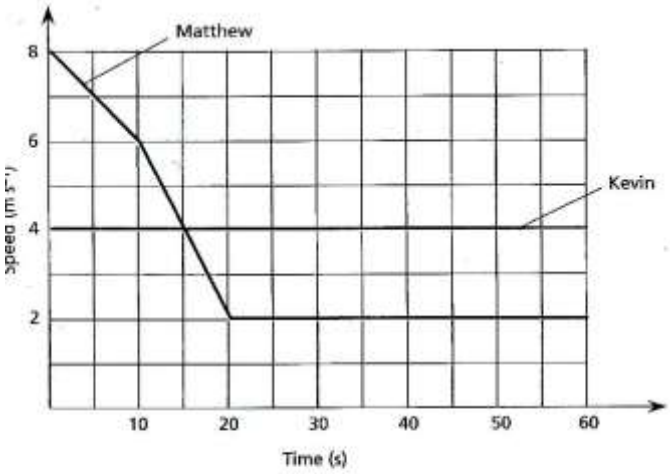
2 A car travelling with a constant speed of 80 km h^{-1} passes a stationary motorcycle policeman. The policeman sets off in pursuit, accelerating to 80 km h^{-1} in 10.0 s and reaching a constant speed of 100 km h^{-1} after a further 5.0 s. At what time will the policeman catch up with the car?

3 A golf ball is dropped from the top of a sheer cliff 78 m above the sea.
a Calculate the velocity of the golf ball after 1.00 s.
b How far will the ball fall in the first 2.00 s?
c Calculate the velocity of the golf ball after it has travelled 10.0 m.
d What is the acceleration of the golf ball at (i) 1.50 s? (ii) 3.50 s?
e When will the golf ball hit the water?

4 A jogger runs along a straight path. Figure 12.40 shows a graph of her velocity against time for the run.
a What is her average speed in the first 80 seconds?
b What distance did she travel in the time shown?
c What is her displacement?
d What is her acceleration at t = 60 s?



5 Two runners are racing over a straight course. Matthew starts off at greater speed but he cannot maintain this rate for the whole race. Kevin runs at a slower but more consistent speed. Their speed–time graphs are shown in Figure 12.41.
a How far does Matthew travel in the first 30 seconds?
b At what time will Kevin pass Matthew?
c If the distance of the race is 240 m, which runner wins the race and by how much?



6 A parachutist is falling vertically with a uniform speed of 12 m s^{-1}. At the instant that he is 50 m above the ground, he drops a coin. (Ignore air resistance.)
a What is the speed with which the coin strikes the ground?
b How long does it take for the coin to reach the ground?
c What is the time difference between the coin and the parachutist reaching the ground?

7. A car, starting from rest, moves with an acceleration of $2ms^{-2}$. Find (a) the velocity at the end of 20 seconds, and (b) the distance covered in that time.

8. With what uniform acceleration does a spacecraft, starting from rest, cover 1000 metres in 10 seconds?

9. A cyclist, starting from rest, moves with an acceleration of $3ms^{-2}$. (a) In what time will it reach a velocity of 30m/s, and (b) what distance does the cyclist cover in this time?

10. An object starts with a velocity of 100m/s and decelerates (slows down) at $2ms^{-2}$.
(a) When will its velocity be zero, and (b) how far will it have gone?

11. A book is knocked off a bench and falls vertically to the floor. If the book takes 1.0s to fall to the floor, calculate:
    a. its speed as it lands.
    b. the height from which it fell.
    c. the distance it falls during the first 0.5s.
    d. the distance it falls during the final 0.5s.

12. A champagne cork travels vertically into the air. It takes 4.0s to return to its starting position.
    a. How long does the cork take to reach its maximum height?
    b. What was the maximum height reached by the cork?
    c. How fast was the cork travelling initially?
    d. What the speed of the cork as it returned to its starting point?
    e. Describe the acceleration of the cork at each of these times after its launch:
        (i) 1.0s (ii) 2.0s (iii) 3.0s.

Answers

1. a(i) 40m (ii) 160m (iii) 160m b 200m c 360m d 13m/s e 0 **2**. 32.5s **3.** a 9.8m/s b 19.6m c 14m/s d(i) $9.8m/s^2$ down (ii) $9.8m/s^2$ up e after 4s 4. a 0.37 m/s b 110 m c -50 m d $-0.017\ m/s^2$

5 a 130m b at t = 35s c Kevin by 50m **6** a 33,8m/s b 2.2s c 2.0s

7. (a) 40 ms^{-1} (b) 400m **8**. 20 ms^{-2} **9**. (a) 10s (b) 150m **10**. (a) 50s (b) 2500m

11. (a) $9.8ms^{-1}$ (b) 4.9m (c) 1.2m (d) 3.7m **12**. (a) 2.0s (b) 19.6m (c) $19.6ms^{-1}$ (d) $19.6ms^{-1}$

(e) (i) $9.8ms^{-2}$ down (ii) $9.8ms^{-2}$ down (iii) $9.8ms^{-2}$ down

Non constant acceleration

2-dimensional representation

Instantaneous velocity can be expressed as

$$\boxed{v = \frac{ds}{dt}}$$

Instantaneous acceleration can be expressed as

$$\boxed{a = \frac{dv}{dt}}$$

Hence,

$$\boxed{a = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2s}{dt^2}}$$

Also, since $v = \frac{ds}{dt}$ and $a = \frac{dv}{dt}$, $vdt = ds$ and $adt = dv$ Hence,

$$\boxed{a\,ds = v\,dv}$$

Note: These rates of change equations aren't restricted to non-constant motion – they can be used for constant motion as well, although the Equations of Motion are commonly used for the latter as the calculations are easier.

Example

An object is moving with a velocity of $v = 3t^2 - 12t$ Calculate its

(a) velocity after 2s, 4s, 6s (b) position after 2s, 4s, 6s (c) acceleration after 2s, 4s, 6s

(d) displacement after 6s, distance after 6s, (e) average velocity after 4s (f) average speed after 6s

(a) $v = 3t^2 - 12t$ When t = 2s $v = 12 - 24 = -12\,\text{m/s}$

When t = 4s $v = 48 - 48 = 0$

When t = 6s $v = 108 - 72 = +36\ \text{m/s}$

(b) $v = \frac{ds}{dt} = 3t^2 - 12t$

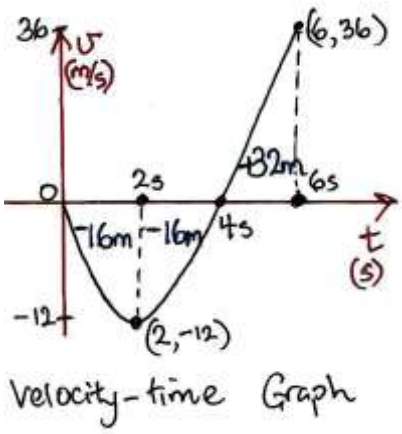
$\int_0^s ds = \int_0^2 3t^2 - 12t\ dt$ $\int_0^s ds = \int_0^4 3t^2 - 12t\ dt$ $\int_0^s ds = \int_0^6 3t^2 - 12t\ dt$

$s = \left[t^3 - 6t^2\right]_0^2 = 8 - 24 = -16\text{m}$ $s = \left[t^3 - 6t^2\right]_0^4 = 64 - 96 = -32\text{m}$ $s = \left[t^3 - 6t^2\right]_0^6 = 216 - 216 = 0$

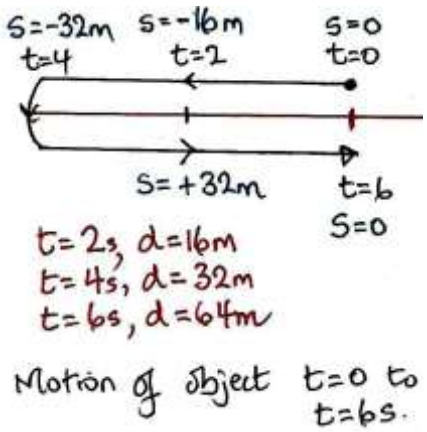
(c) $v = 3t^2 - 12t$, $a = \frac{dv}{dt} = 6t - 12$

$a_{2s} = 6(2) - 12 = 0$ $a_{4s} = 6(4) - 12 = 12\,\text{m/s}^2$ $a_{6s} = 6(6) - 12 = 24\,\text{m/s}^2$

Sketch the *v-t* graph, and label the area (displacement) under the curve (part (b)).



Motion of the object t = 0 – 6s



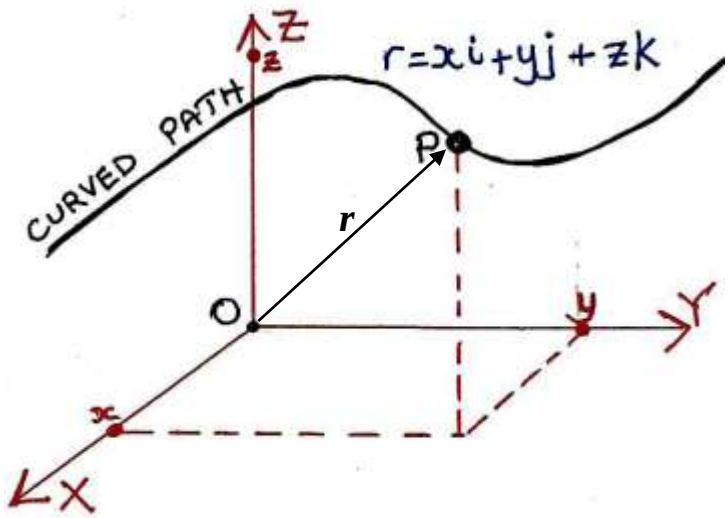
Displacement = 0, see also part (b);
Distance = 64m

(e) Average Velocity $v_{AV} = \frac{s}{t} = \frac{-32}{4} = -8\,\text{m/s}$

(f) Average Speed $|v_{AV}| = \frac{d}{t} = \frac{64}{6} = 10.7\,\text{m/s}$

3-dimensional representation

Consider the curved path – often called a space curve – in 3-dimensional space shown in the diagram.



A point ***P*** on this path is given by the position vector

$$\boxed{\boldsymbol{OP} = \boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}}$$

and its magnitude by $|\boldsymbol{r}| = \sqrt{(x)^2 + (y)^2 + (z)^2}$

Differentiation of $\boldsymbol{r}$ yields the velocity, or

$$\boxed{\frac{dr}{dt} = \boldsymbol{v} = v_x\boldsymbol{i} + v_y\boldsymbol{j} + v_z\boldsymbol{k}}$$

which is a tangent to the curve at point *P*, and its magnitude (speed) is given by

$$|\boldsymbol{v}| = \sqrt{(v_x)^2 + (v_y)^2 + (v_z)^2}$$

Similarly, Differentiation of v yields the acceleration, or

$$\boxed{\frac{dv}{dt} = \boldsymbol{a} = a_x\boldsymbol{i} + a_y\boldsymbol{j} + a_z\boldsymbol{k}}$$

and its magnitude is given by

$$|\boldsymbol{a}| = \sqrt{(a_x)^2 + (a_y)^2 + (a_z)^2}$$

Example

A particle *P* is travelling along the path $\boldsymbol{r} = (3\cos t)i + (2\sin t)j + tk$ metres, $t \geq 0$. Calculate, in exact form

(a) the position vector of *P* at $t = \frac{\pi}{3}$ seconds (b) the velocity vector of *P* at $t = \frac{\pi}{3}$ seconds

(c) the speed of *P* at $t = \frac{\pi}{3}$ seconds (d) the acceleration vector of *P* at $t = \frac{\pi}{3}$ seconds

Solution

(a) Position $\boldsymbol{r} = (3\cos t)i + (2\sin t)j + tk$ Substituting $t = \frac{\pi}{3}$

$$\boldsymbol{r} = \left(3\cos\frac{\pi}{3}\right)i + \left(2\sin\frac{\pi}{3}\right)j + \frac{\pi}{3}k$$

$$= \left(\frac{3}{2}i + \sqrt{3}j + \frac{\pi}{3}k\right) \text{ m}$$

(b) Velocity $\boldsymbol{r} = (3\cos t)i + (2\sin t)j + tk$. Differentiate and substitute $t = \frac{\pi}{3}$

$$\boldsymbol{v} = \frac{dr}{dt} = (-3\sin t)i + (2\cos t)j + k$$

$$= \left(-3\sin\frac{\pi}{3}\right)i + \left(2\cos\frac{\pi}{3}\right)j + k = \left(\frac{-3\sqrt{3}}{2}i + j + k\right) \text{ m/s}$$

(c) Speed $|\boldsymbol{v}| = \sqrt{(v_x)^2 + (v_y)^2 + (v_z)^2}$ = $\sqrt{\left(\frac{-3\sqrt{3}}{2}\right)^2 + 1^2 + 1^2} = 2.95$ m/s

(d) Acceleration $\boldsymbol{v} = (-3\sin t)i + \left(2\cos\frac{\pi}{3}\right)j + k$ and substitute $t = \frac{\pi}{3}$

$$\boldsymbol{a} = \frac{dv}{dt} = (-3\cos t)i + (-2\sin t)j + 0$$

$$= \left(-3\cos\frac{\pi}{3}\right)i + \left(-2\sin\frac{\pi}{3}\right)j = \left(-\frac{3}{2}i - \sqrt{3}j\right) \text{ m/s}^2$$

Exercise

1 A particle is moving with a velocity of $v = -3t^2 + 6t$ where v is its velocity in m/s. Calculate its
- a velocity after 2s, 4s, 6s
- b position after 2s, 4s, 6s
- c acceleration after 2s, 4s, 6s
- d distance after 6s
- e average velocity after 6s
- f average speed after 6s

2 An object follows a path with an acceleration of $a = \frac{1}{2v}$ where v is in m/s. Calculate the object's
- a Velocity after 21 seconds
- b Position after 21 seconds

3 The position of a car is given by $s = 3t^2 - 6t - 9$ where s is in metres and t is in seconds. It starts from rest. Calculate
- a Its displacement after 0, 2s, 4s, 6s
- b Its velocity after 0, 2s, 4s, 6s
- c Its acceleration after 0, 2s, 4s, 6s
- d The total distance travelled after 6 seconds.

4 An object is moving along a straight line with an acceleration $a = 12t^2 - 2$ m/s². When t = 0, the object is at the origin, when t = 2s the object is 16 m to the right of the origin. Where is the object after 3 seconds?

5 A charged particle is following a straight line path with a velocity of $v = 40 - 0.1s^2$ m/s.
- a Calculate the acceleration of the particle when s = 10m
- b Describe the particle's motion as it moves from s = 0 to s = 10m to s = 20m to s = 30m

6 A particle P is travelling along the path $\boldsymbol{r} = (t^2 + 3t)i + (t^3)j + (t-1)k$ metres, $t \geq 0$. Calculate
- a Its position vector at $t = 0$
- b Its velocity vector at $t = 0$
- c Its acceleration vector at $t = 0$
- d Its speed at $t = 1$ s
- e Its displacement from $t = 2$ to $t = 3$ seconds
- f Its change in velocity from $t = 2$ to $t = 3$ seconds
- g Its average speed from $t = 2$ to $t = 3$ seconds

7 A particle P is travelling along the path $\boldsymbol{r} = (2\sin t)i + (3\cos t)j + t^3 k$ metres, $t \geq 0$. Calculate, in exact form
- a the position vector of P at $t = \frac{\pi}{6}$ seconds
- b the velocity vector of P at $t = \frac{\pi}{6}$ seconds
- c the speed of P at $t = 0$ seconds
- d the acceleration vector of P at $t = \frac{\pi}{6}$ seconds
- e The time at which the x-component of the particle's velocity is $1.5i$ m/s

8 A particle P is travelling along the path $\boldsymbol{r} = (1+t)i + \left(\frac{t^2}{\sqrt{2}}\right)j + \left(\frac{t^3}{3}\right)k$ metres, $t \geq 0$.

Calculate
- a the particle's velocity, in terms of t
- b the particle's acceleration, in terms of t
- c the particle's speed at $t = 1$s

Answers

1 a 0; -24; -72 m/s
b 4; -16; -108 m
c -8; -24; -40 m/s^2
d 116 m
e -12 m/s
f 19.2 m/s

2 a 11 m/s
b 887.3 m

3 a -9; -9; 15; 63 m
b -6; 6; 18; 30 m/s
c 6m/s^2 at all times
d 78 m

4 78 metres to the right of the origin

5 a -60 m/s^2
b s = 0, v = 40m/s; s = 10m, v = 30m/s; s = 20m, v = 0; s = 30m, v = -50m/s. Particle is travelling at 40m/s in a positive direction at s = 0 (origin). It then decelerates to a stop by s = 20m. By s = 30m, the particle is travelling in an opposite direction at 50m/s.

6 a $\boldsymbol{s} = -k$ m
b $\boldsymbol{v} = 3i$ m/s
c $\boldsymbol{a} = 2i$ m/s^2
d $|\boldsymbol{v}| = 5.8$ m/s
e $\boldsymbol{r} = 8i + 19j + k$
f $\Delta\boldsymbol{v} = 2i + 15j$
g $|\boldsymbol{v}| = 15.1$ m/s

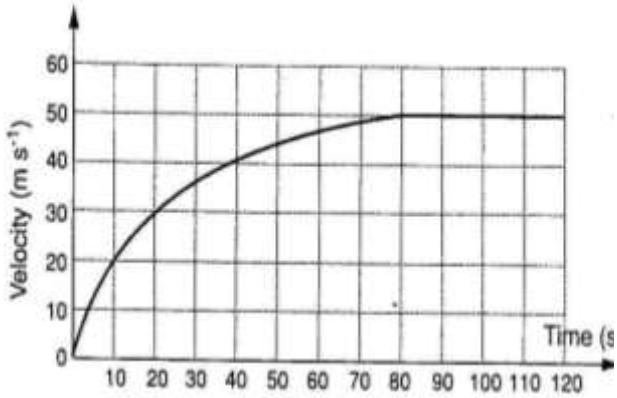
7 a
$$\boldsymbol{r} = \frac{1}{2}i + \frac{3\sqrt{3}}{2}j + \frac{\pi^3}{216}k \quad \text{metres}$$
b
$$\boldsymbol{v} = \sqrt{3}\,i + \frac{3}{2}j + \frac{\pi^2}{12}k \quad \text{m/s}$$
c $|\boldsymbol{v}_{AV}| = 2$ m/s
d
$$\boldsymbol{a} = -i + \frac{3\sqrt{3}}{2}j + 2\pi k$$

8 a $\boldsymbol{v} = i + \sqrt{2}\,tj + t^2k$
b $\boldsymbol{a} = \sqrt{2}\,j + 2tk$
c $|\boldsymbol{v}| = 2$ m/s

Additional Questions

g = 9.8m/s^2 Answers to 1 decimal place

1 The straight line motion of a high-speed train is shown below.
   (a) How long does it take the train to reach its cruising speed?
   (b) What is the acceleration of the train 10s after starting?
   (c) What is the acceleration of the train 40s after starting?
   (d) What is the displacement of the train after 120s?



2 A stone is thrown is thrown vertically upwards with a velocity of 10.0m/s from the edge of a cliff 65m high.
   (a) How much later does it reach the bottom of the cliff?
   (b) What is its speed just before hitting the ground?
   (c) What total distance did the stone travel?

3 A particle P is projected from the origin 0 so that it moves along the *x*-axis. At time t s after projection, the velocity of the particle, v ms^{-1}, is given by $v = 3t^2 - 22t - 45$
   (a) Calculate the acceleration of the particle at t = 2s.
   (b) Calculate the displacement of the particle at t = 2s.
   (c) Find the displacement of the particle at t = 10s.
   (d) Find the distance travelled by the particle at t = 10s.

4 A particle P is travelling along the path $\boldsymbol{r} = (2\sin 3t)i + (3\cos 2t)j + tk$ metres, $t \geq 0s$.
Calculate, in exact form
   (a) the position vector of P at $t = \frac{\pi}{6}$ seconds
   (b) the velocity vector of P at $t = \frac{\pi}{6}$ seconds
   (c) the speed of P at $t = 0$ seconds
   (d) the acceleration vector of P at $t = \frac{\pi}{6}$ seconds

Answers

1 (a) 80s (b) ~1.3m/s^2 (c) ~0.5m/s^2 (d) ~4900m

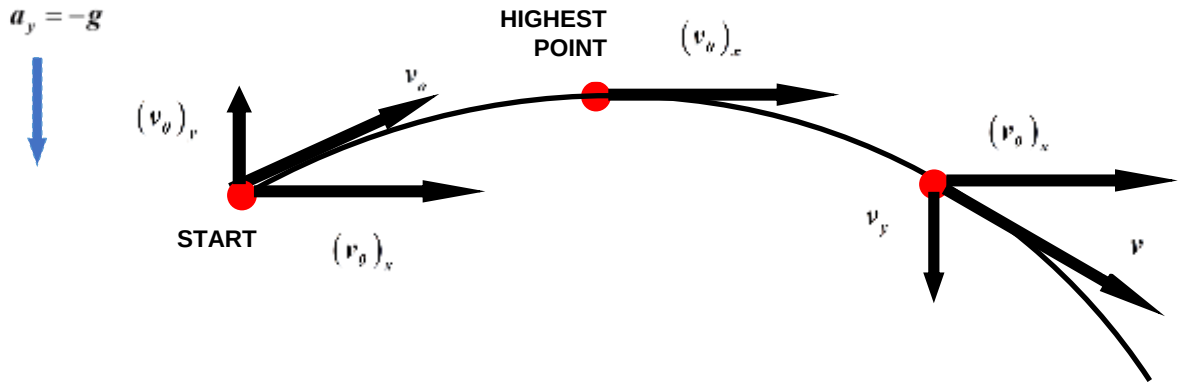
2 (a) 4.8s (b) 37m/s (c) 75m

3 (a) -10m/s^2 (b) -126m (c) -550m (d) 584m

4 (a) $\boldsymbol{r} = \left(2i + \frac{3}{2}j + \frac{\pi}{6}k\right)$m (b) $\boldsymbol{v} = \left(-3\sqrt{3}j + k\right)$m/s (c) $|\boldsymbol{v}| = \sqrt{37}$m/s (d) 0

PROJECTILE and ANGULAR MOTION

Projectile motion



Horizontal Motion $a_x = 0$	Vertical Motion $a_y = -g$
$v = (v_0)_x$	$v = (v_0)_y + a_y t = (v_0)_y - gt$
$s = (v_0)_x t$	$v^2 = (v_0)_y^{\ 2} + 2a_y s_y = (v_0)_y^{\ 2} - 2gs_y$
$a = 0$	$s = (v_0)_y t + \frac{1}{2}a_y t^2 = (v_0)_y t - \frac{1}{2}gt^2$

Example An object is projected at an angle of 60° to the horizontal with a velocity of 25m/s. Calculate

a The maximum height reached by the projectile

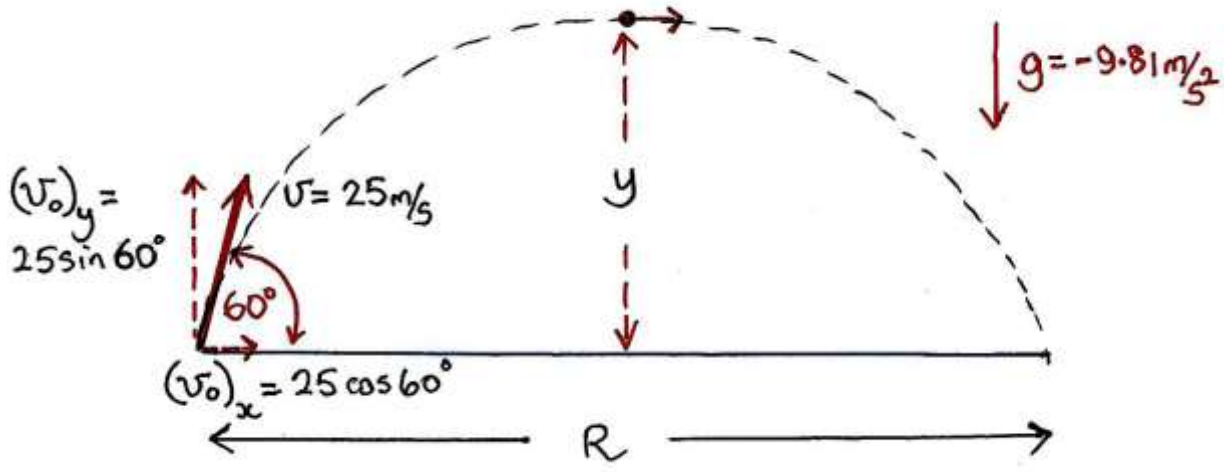
b The time taken by the projectile to reach its maximum height

c The total time taken by the object throughout its flight

d The velocity of the projectile when at its maximum height

e The horizontal distance travelled by the projectile

f The velocity of the projectile at t =3s Give its magnitude and direction



a List the data. ***Vertical component only***.

$y = ?\quad (v_0)_y = 25\sin 60 = 21.7 \quad a = g = -9.81 \quad v = 0$

$v^2 = v_0^{\,2} + 2as \qquad 0^2 = 21.7^2 + 2(-9.81)s, \quad s = \dfrac{468.7}{19.62} = 23.9\ \text{m/s}$

b List the data. ***Vertical component only***.

$t = ?\quad (v_0)_y = 21.7 \quad a = g = -9.81 \quad v = 0$

$v = v_0 + at \qquad 0 = 21.7 + (-9.81)t, \quad t = \dfrac{21.7}{9.81} = 2.2\text{s}$

c Given the symmetry of the flight (parabola), if it takes 2.2s to reach its maximum height, then the total time of flight will be $2 \times 2.2 = 4.4$s

d Note that, due to gravitational acceleration, the ***vertical component*** of the object's velocity decreases as it rises to its maximum height. At this point the vertical velocity component is zero. Since there is no acceleration acting on the projectile in the ***horizontal direction***, the horizontal velocity component of the projectile will be constant. Answer: $(v_0)_x = 25\cos 60 = 12.5\text{m/s}$

e As per part (d), the projectile's velocity in the horizontal direction is constant at 12.5m/s. The horizontal distance travelled *R* (commonly called the ***Range***) is therefore

$(v_0)_x = \dfrac{R}{t}, \quad \text{or} \quad R = (v_0)_x \times t = 12.5 \times 4.4 = 55\text{m}$

f ***Vertical component***

$v = ?\quad t = 3;\ (v_0)_y = 21.7 \quad a = g = -9.81$

$v = v_0 + at \qquad v = 21.7 + (-9.81)(3) = -7.73\text{m/s}$

Note that the velocity is negative indicating that the projectile's vertical direction at *t* = 3s is downwards

Horizontal component

$(v_0)_x = 12.5\text{m/s}$

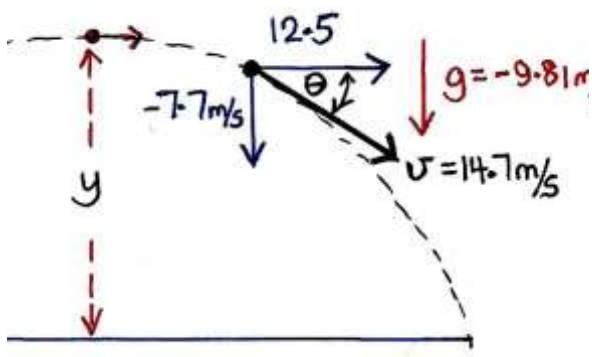
Note that the horizontal component of the projectile's velocity is constant throughout its flight.

To find the velocity of the projectile at *t* = 3s, construct a vector triangle and use Pythagoras and trigonometry.

$v = \sqrt{12.5^2 + (-7.7)^2} = 14.7\text{m/s}$

$\theta = \tan^{-1}\left(\dfrac{7.7}{12.5}\right) = 31.6°$

The projectile's velocity is 14.7m/s at an angle of 31.6° below the horizontal.



Exercise

Give answers to 1 decimal point. Use $g = 9.81\ ms^{-2}$

1 A rock is thrown horizontally with a velocity of 30m/s from the top of a cliff 80m high.

a How long does it take the rock to fall to the ground

b What horizontal distance from the base of the cliff does the rock land?

c What is the velocity of the rock just before it hits the ground? Magnitude and direction required. 49.4m/s

2 A plane is flying horizontally at a speed of 100m/s and drops a marker buoy that lands in the water 4km from where it was dropped from the plane..

a How long did it take the marker buoy to fall to the water?

b What was the altitude of the plane, in km?

3 An object is projected at an angle of 30° to the horizontal with a velocity of 30m/s. Calculate

a The maximum height reached by the projectile

b The time taken by the projectile to reach its maximum height

c The total time taken by the object in its flight

d The velocity at the maximum height reached by the projectile

e The horizontal distance travelled by the projectile

f The velocity of the projectile at t =3s. Give its magnitude and direction.

4 A boy is standing on a ledge 5m high and throws a ball with a velocity of 20m/s at an angle of 37°. Calculate the horizontal distance travelled by the ball.

5. A basketball player shoots for goal. The ball, 2.1m above the ground, leaves her hand at an angle of 35° and she wants to drop the ball into the ring at the highest point in its trajectory. The ring is 2.6m above the ground. What speed must the ball leave her hand?

6 The range of a projectile R is the horizontal distance travelled before returning to its original height when projected at an angle θ with an initial velocity v_0. Show that the range is given by

$$R = \frac{v_0^{\ 2} \sin 2\theta}{g}$$

Answers

1 (a) 4.0s (b) 120m (c) 52.6° below the horizontal

2 (a) 40s (b) 7.84km

3 (a) 11.5m (b) 1.5s (c) 3.0s (d) 26m/s (e) 78m (f) 29.7m/s at 29.0° below the horizontal

4 30.7m

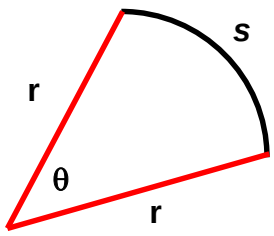
5 5.5m

Angular motion

Arc Length s (m)

Generally, the angular displacement θ at the centre of a circle is defined in radians by the equation

$$\theta = \frac{\text{arc length}}{\text{radius}} = \frac{s}{r} \quad \text{or} \quad \boxed{s = r\,\theta}$$



In a complete circle

$$360^\circ = 2\pi \text{ radians} \quad \text{or} \quad \pi \text{ radians} = 180^o$$

Conversions

From radians to degrees: $\times \dfrac{180}{\pi}$ From degrees to radians $\times \dfrac{\pi}{180}$

Angular velocity ω (rad/s)

Angular Velocity ω is the rate of change of angular displacement

$$\boxed{\omega = \frac{\theta}{t} \quad \text{or} \quad \omega = \frac{d\theta}{dt}}$$

Linear and angular velocity are related by the formula

$$\boxed{v = r\,\omega}$$

Revolutions per Minute rpm

Since 1 revolution $= 2\pi$ radians , I revolution per minute (60 seconds) is $\dfrac{2\pi}{60} = \dfrac{\pi}{30}$ rad/s

Conversions: From rpm to radians per second $\times \dfrac{\pi}{30}$ From radians per second to rpm $\times \dfrac{30}{\pi}$

Example A weight on the end of a string describes a circular path of radius 0.5m. If its linear velocity is 2m/s what is the angular velocity in (a) rad/s (b) rpm?

(a) $\omega = \dfrac{v}{r} = \dfrac{2}{0.5} = 4$ rad/s (b) $4 \text{ rad/s} = 4 \times \dfrac{30}{\pi} = 38.2 \text{ rpm}$

Angular Acceleration α (rad/s^2)

Angular acceleration is the rate of change of angular velocity

$$\boxed{\alpha = \frac{\omega}{t} \quad \text{or} \quad \alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}}$$

Linear and angular acceleration are related by the formula

$$\boxed{a = r\,\alpha} \quad \text{or} \quad \boxed{a = \omega^2 r}$$

Uniform acceleration: Equations of Angular Motion

It can be shown that equations derived for linear motion apply equally to angular motion

Linear Equations	*Angular Equations*
$s = vt$	$\theta = \omega t$
$a = \dfrac{v - v_0}{t}$	$\alpha = \dfrac{\omega - \omega_0}{t}$
$v = v_0 + at$	$\omega = \omega_0 + \alpha t$
$v^2 = v_0^2 + 2as$	$\omega^2 = \omega_0^2 + 2\alpha\theta$
$s = v_0 t + \frac{1}{2}at^2$	$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$

Example A flywheel increases in velocity from rest to 400 rpm. in 15 seconds. Calculate (a) the angular acceleration, and (b) the number of revolutions it makes in this time.

(a) $\omega_0 = 0,\ \omega = 400\text{rpm} = \dfrac{400 \times \pi}{30} = 41.9\ \text{rad/s},\quad t = 15,\quad a = ?$

Use equation $\omega = \omega_0 + \alpha t,$ $\alpha = \dfrac{\omega - \omega_0}{t} = \dfrac{41.9 - 0}{15} = 2.79\,\text{rad/s}^2$

(b) $\theta = ?$ Use equation $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$

$\theta = \omega_0 t + \frac{1}{2}\alpha t^2 = 0(15) + \frac{1}{2}(2.79)(15)^2 = 314\ \text{radians} = \dfrac{314}{2\pi} = 50\ \text{revs}$

Non-uniform acceleration

Example A flywheel of radius 0.5m rotates at an angular velocity of $\omega = \dfrac{4}{\theta}$ rad/s. Given that the flywheel's acceleration is non-uniform, calculate

a the flywheel's angular displacement when its angular velocity is 2 rad/s

b the time it takes to reach this angular velocity of 2 rad/s. When t = 0, $\theta = 0$

c the acceleration of the flywheel when the angular velocity is 2 rad/s.

a $\omega = \dfrac{4}{\theta},\quad \theta = \dfrac{4}{\omega}$ Substituting : $\theta = \dfrac{4}{\omega} = \dfrac{4}{2} = 2$ radians

b $\omega = \dfrac{d\theta}{dt}$, $dt = = \dfrac{d\theta}{\omega}$. Integrating both sides $\int dt = \int \dfrac{d\theta}{\omega}$

$$t = \int \frac{\theta}{4} d\theta = \frac{\theta^2}{8} + c$$

When t = 0. $\theta = 0$, or c = 0. Substituting: $t = \dfrac{\theta^2}{4} = \dfrac{2^2}{4} = 1.0$

The flywheel takes 1.0s to reach an angular velocity of 2 rad/s

c $\omega = \frac{4}{\theta} = 4\theta^{-1}$; $\frac{d\omega}{dt} = -4\left(\theta^{-2}\right)\frac{d\theta}{dt}$ using the Chain Rule $\frac{d\omega}{dt} = \frac{d\omega}{d\theta} \times \frac{d\theta}{dt}$

Substituting for θ and ω

$a = \frac{d\omega}{dt} = -4(2^{-2})(2) = -2$ $a = r\alpha = 0.5 \times -2 = -1.0$ m/s²

The flywheel is decelerating at -1.0 m/s² when its angular velocity is 2 rad/s

Exercise

1 The bob of a pendulum of length 1.5m swings through a 20cm arc. Calculate the angular displacement in (a) radians (b) degrees.

2 The flywheel of a motor is revolving at 1500 rpm and slows down uniformly to 1200 rpm in 15 seconds. Calculate (a) the initial and final angular velocities in rad/s (b) the angular acceleration and (c) the linear acceleration of a point 1.5m from the centre (d) the angular distance travelled in radians and revolutions

3 A flywheel operates at 300 r.p.m. Calculate (a) the angular velocity and (b) the linear velocity 1.25 from the centre.

4 Calculate (a) the angular velocity of a car which rounds a curve of radius 10m at 60 km per hr.

5 A laser beam, directed on to the moon, 4.00×10^{8}m from the Earth, subtends an angle of 2.00×10^{-4} rad. Calculate the diameter of the circular image that the laser beam makes on the moon.

6 Calculate (a) the tangential velocity and (b) the angular velocity of the Earth at the equator if the radius of the Earth at the Equator is 6300 km. Assume that the Earth is spherical.

7 A flywheel of radius 0.1m rotates with an angular velocity of $\omega = \frac{100}{\theta}$, where θ is in radians.

Calculate

a The angular distance travelled after the flywheel reaches an angular velocity of 50 rad/s from a stationary start.

b The time it takes to reach this angular velocity of 50 rad/s

c The acceleration of the flywheel on reaching an angular velocity of 50 rad/s.

8 A flywheel, of radius 0.1m, starts from rest and rotates at an angular acceleration of $\alpha = 10e^{-t}$ rad s^{-2} . Calculate

a Its angular velocity after 10s

b Its speed after 10s

c Its angular displacement and the revolutions made after 10 seconds

Answers

1 (b) 0.13 rad (b) 7.5°

2 (1) 157.1 rad/s; 125.7 rad/s (b) 6.28 rad s^{-2} (c) 9.4 m s^{-2}(d) 1650 radians; 263 revs

3 (1) 31.42 rad s^{-2} (b) 39.28 m s^{-2}

4 1.7 rad s^{-1}

5 4.00×10^{8}m

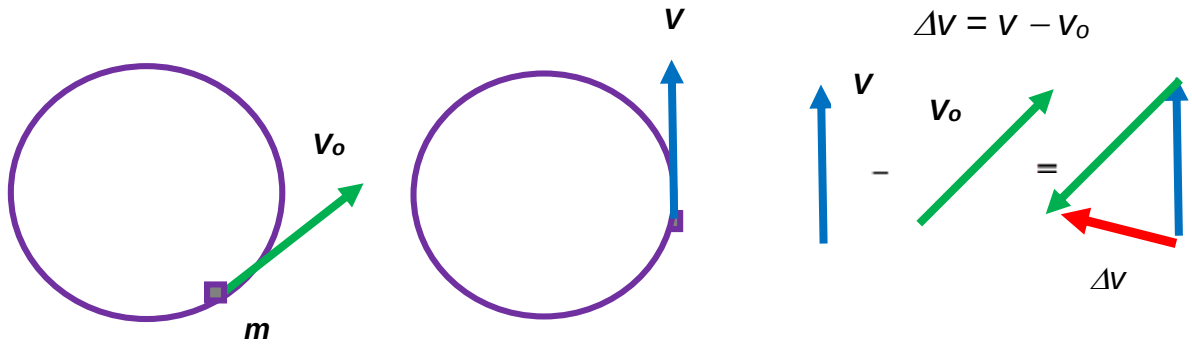
6 (a) 458m/s (b) 7.3×10^{-5} rad/s

7 (a) 2 rad (b) 0.02 s (c) -2.5 m s^{-2}

8 (a) 10.0 rad/s (b) 1.0m/s (c) 90 rad; 14.3 revs

Radial (centripetal) acceleration a_r

Consider a mass *m* travelling at a ***uniform speed*** in a horizontal circle. In the position shown below left the instantaneous velocity is shown as a vector v_0



If we look at this mass a short time later it will look like that shown in the diagram above centre. The instantaneous velocity is now *v* . Thus, the change in velocity Δv, shown above right, is given by

$$\Delta v = v - v_o$$

Note that Δv must always act radially inwards. Therefore, the mass is accelerating even though the speed is not changing. This acceleration is called radial (centripetal) acceleration**,** indicating that the acceleration always acts towards the centre of the circle**.** Radial acceleration describes the change in velocity with time for an object moving in a circular path at constant speed. The acceleration is given by:

$$\boxed{a_r = \frac{v^2}{r} = \omega^2 r}$$

$v = r \times \omega$ from angular motion. a_r = radial acceleration (ms^{-2}), v = speed (m/s). r = radius of circle (m),

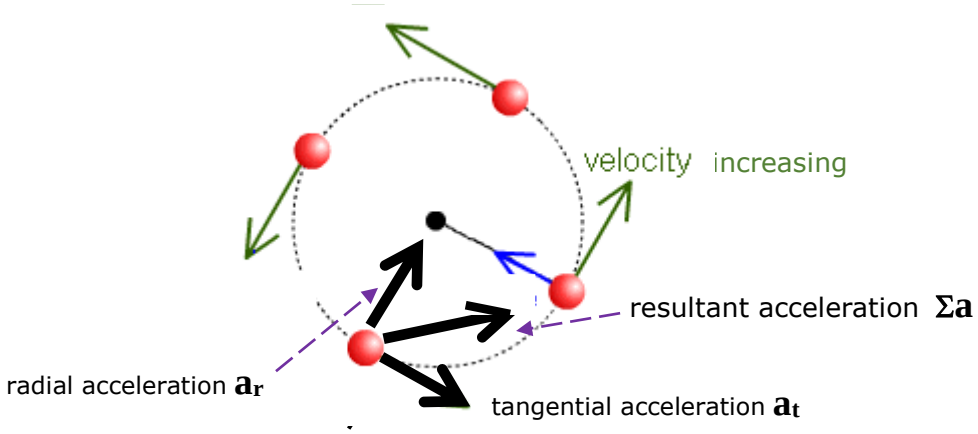
Radial force F_r

The net force on an object is given by $\Sigma F = m \times a$, where ΣF is always in the same direction as the acceleration *a*. Since the acceleration is radial, the net force also acts towards the centre of the circle. This net force is called the radial (or centripetal) force $\mathbf{F_R}$**.** Since $\Sigma F = m \times a$

$$\boxed{F_r = ma = \frac{m v^2}{r} = m\omega^2 r}$$

Non-uniform circular motion

A tangential force $\mathbf{F_t}$ gives rise to a tangential acceleration $\mathbf{a_t}$ acting on an object travelling in a circle which gives rise to non-uniform circular motion. There will still be a radial acceleration $\mathbf{a_r}$ that acts towards the centre of the circle (see below). The radial and tangential accelerations will be perpendicular to each other. The resultant vector acceleration $\mathbf{\Sigma a}$ of $\mathbf{a_t}$ and $\mathbf{a_r}$ is given by $\mathbf{\Sigma a} = \mathbf{a_t} + \mathbf{a_r}$



Forces that cause circular motion

A car moving in a circle does so because the radial force is due to the friction between the tyres and the road. The radial force causing a satellite, such as the Earth's natural satellite the moon, to orbit the Earth is caused by the gravitational force of the Earth acting on the satellite. An object at the end of a string being swung in a circle will have a tensional force acting towards the centre of the circle caused by the person pulling inwards on the string.

Example: Uniform Acceleration

An athlete swings a 2.5kg hammer in a horizontal circle of radius 0.8m at 2.0 revolutions per second. The wire is horizontal at all times and the hammer is travelling at a constant speed..

(a) What is the orbital speed of the ball?
(b) Calculate the acceleration of the ball?
(c) What is the magnitude of the net force acting on the ball?
(d) Name the force that is responsible for the radial acceleration of the ball?
(e) Describe the motion of the ball if the wire breaks?

(a) $v = (2\times\pi\times r)/t$ where t is the time for 1 rev. 2 revs/s = 1 rev/0.5s. Hence, $v = 2\pi(0.8)/0.5 = 10.1$ m s^{-1}
(b) $a = v^2/r$
$= (10.1\text{m s}^{-1})^2/0.80\text{ m} = 126\text{ m s}^{-2}$ towards centre of circle
(c) $F = ma$
$= (2.5\text{ kg})(126\text{ m s}^{-2}) = 316$ N
(d) The force that is responsible for the radial acceleration of the ball is the tension in the wire caused by the athlete pulling on the hammer. The acceleration is directed radially inwards at all times.
(e) If the wire breaks, the ball will move off at a tangent to the circle with a speed of 10.1 m s^{-1}.

Example: Non-Uniform Acceleration

A flywheel, of radius 0.1m, is initially at rest. It then starts up and rotates with an acceleration of $a = 6t$ ms^{-2}, where t is in seconds. Calculate

a The angular acceleration of the flywheel after rotating for 5 seconds.
b The angular velocity of the flywheel in rad/s after rotating for 5 seconds.
c The angular velocity of the flywheel in revs/min after rotating for 5 seconds.
d The angular displacement θ of the flywheel after rotating for 5 seconds.
e The radial acceleration of the flywheel after rotating for 5 seconds.
f The time it takes for the flywheel to rotate 10 000 revolutions.

a $a = r\alpha$, where $a = 6t$ and r = 0.1m. Hence, $\alpha = \frac{a}{r} = \frac{6t}{0.1} = 60t$

When t = 5s, $\alpha = 60(5) = 300$ rad/s^2.

b $\alpha = \frac{d\omega}{dt} = 60t,\ d\omega = \alpha\, dt$ Hence $\omega = \int 60t\, dt = 30t^2 + c$

When t = 0, $\omega = 0$ so c = 0. Hence $\omega = 30t^2$. Substituting for t: $\omega = 30(5)^2 = 750$ rad/s

c $750\text{ rad/s} = 750\times\frac{30}{\pi} = 7162\text{ rev/min}$

d $\omega = \frac{d\theta}{dt} = 30t^2$

So $d\theta = 30t^2\, dt$, $\theta = \int 30t^2\, dt = 10t^3 + c$

When t = 0, $\theta = 0$ so c = 0. Hence $\theta = 10t^3$. Substituting for t: $\theta = 10(5)^3 = 1250$ rad

e Radial acceleration is given by $a_r = \omega^2 r$. Substituting for ω and r

$a_r = \omega^2 r = 750^2(0.1) = 56250\text{ rad/s}^2$

f $1\text{ rev} = 360^o = 2\pi$ radians Hence 10 000 revs = 20000π radians $= 62831$ radians

Since $\theta = 10t^3$, $t = \sqrt[3]{\frac{62831}{10}} = 18.4$ s

Exercise

1 An ice skater of mass 50 kg is skating in a horizontal circle of radius 1.5m at a constant speed of 2.0 m/s.
   a What is the acceleration of the skater?
   b What is the horizontal component of the radial force acting on the skater?
   c Name the force that is providing the horizontal component of the net force enabling the skater to move in a circular path?

2 A radial force of 30N is applied to a 2kg object to allow it to rotate uniformly in a horizontal circle of radius 1.5m.
   a Calculate the radial acceleration of the object
   b Calculate the velocity of the object
   c Calculate the angular velocity of the object.
   d Calculate the frequency of the object.

3 A 200g tennis ball at the end of a string is swinging in a horizontal circle of radius 2m. The ball rotates at 3 revolutions per second.
   a How far has the tennis ball travelled after 3 revolutions?
   b Calculate the velocity of the tennis ball?
   c Calculate the angular velocity of the tennis ball?
   d Calculate the radial acceleration of the ball?
   e Calculate the frequency of the ball

4 The moon makes one orbit of the Earth in 28 days. Assuming the orbit is circular and the radius of the centre of the moon to the centre of the Earth is 4.00×10^{8}m, calculate
   a The velocity of the moon in its orbit.
   b The radial acceleration of the moon towards the Earth.
   c The angular velocity of the moon
   d The radial force of the Earth acting on the moon if the moon's mass is 7.0×10^{22}kg

5 A rope wound around a large wheel, of radius 2.0m, is unwinding a block, of mass 2kg, which is descending vertically to the ground. Its angular displacement is $\theta = 0.2t^3 + 10t$, where t Is in seconds. Calculate
   a The velocity of the block after 5 seconds.
   b The acceleration of the block after 5 seconds.
   c The net force acting on the block after 5 seconds
   d The radial acceleration of the wheel after 5 seconds.

Answers

1 (a) 2.7 m s^{-2} towards centre of circle
(b) towards centre of circle (c) friction

2 (a) 15 m s^{-2} (b) 4.7 m/s (c) 3.1 rad/s (d) 0.5Hz or 0.5 revs/s

3 (a) 37.7m (b) 12.6m/s (c) 6.3 rad/s (d) 79.0 m s^{-2} (e) 1.0Hz or 1.0 revs/s

4 (a) 1038m/s (b) 2.7×10^{-3} m s^{-2} (c) 2.6×10^{-6} rad/s (d) 1.9×10^{20} N

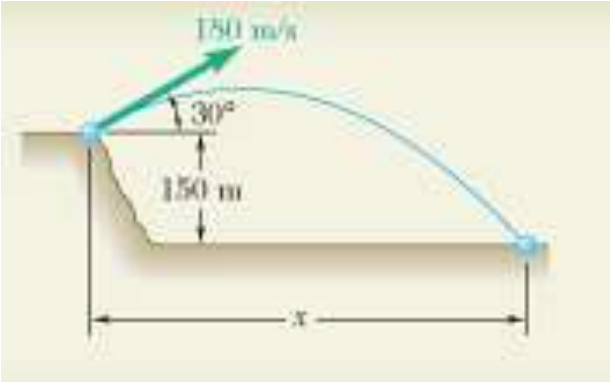
5 (a) 50m/s (b) 12 m s^{-2} (c) 24N (d) 1250 m s^{-2}

Additional Questions

g = 9.8m/s^2 Answers to 1 decimal place

1 A projectile is fired from the edge of a 150m cliff with an initial velocity of 180m/s at an angle of 30° with the horizontal. Neglecting air resistance caluclate

(a) the time of flight
(b) the horizontal distance (x) from the gun to the point where the projectile strikes the ground
(c) the greatest elevation above the ground reached by the projectile.



2 The wheels of a car make 70 revolutions as the car reduces its speed uniformly from 90km/h to 50km/h. The wheels have a diameter of 1.0m.
(a) What was the angular acceleration of the wheels?
(b) If the car continues to decelerate at this rate how much time is required for the car to stop?

3 A cord is wrapped around a wheel, of diameter 0.2m, initially at rest. If a force is applied to the cord and gives it an acceleration of $a = 4t$ m/s^2 , calculate
(a) the angular velocity of the wheel after 2 seconds.
(b) the angular displacement of the wheel after 4 seconds.

4 What centripetal force is exerted on the Moon as it orbits about Earth at a centre-to-centre distance of 3.84×10^8 m with a period of 27.4 days? The mass of the Moon is 7.35×10^{22} kg.

Answers

1 (a) 19.9s (b) 3102.9m (c) 563.3m

2 (a) -2.0 rad/s^2 (b) 25.0s

3 (a) 40.0 rad/s (b) 213.3 rad

4 2.0×10^{22} N