ENST1.2: MOMENT OF INERTIA

Mass Moment of Inertia

Net force, mass and acceleration in linear motion have a rotational analogue – net torque, mass moment of inertia and angular acceleration respectively. Mass moment of inertia is a measure of the resistance offered by an object to rotational movement, e.g. the bending moment in a beam.

Linear Motion

$$\Sigma F = m \times a$$

Rotational Motion

$$\Sigma \tau = I \times \alpha$$

where $\Sigma\tau$ = net torque (Nm); I = mass moment of inertia (kgm^2); α = angular acceleration ($rad\ s^{-2}$).

Example

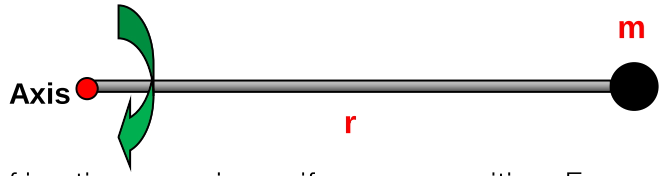
Calculate the mass moment of inertia of a flywheel that has a net torque of 15 Nm and an angular acceleration of 25 $rads^{-2}$

Solution

$$\Sigma\tau = I \times \alpha \Rightarrow I = \frac{\Sigma\tau}{\alpha} = \frac{15}{25} = 0.6\,kgms^{-2}$$

The mass moment of inertia I of a rigid body m about an axis of rotation r is

$$I = m \times r^2$$



Different objects have different moments of inertia, assuming uniform composition. For a flywheel (disc) $I = ½\ mr^2$, a cylinder $I = ½\ mr^2$, a thin hoop $I = mr^2$, and a rod with its axis of rotation at one end $I = ⅓\ mr^2$.

(Giancoli, G.C, 1991, *Physics: principles with applications*, Prentice-Hall)

Radius of gyration

Notice above that a thin hoop has the same moment of inertia as the general formula $I = mr^2$. With the hoop all the mass is concentrated at the same distance from the axis. The radius of gyration is a similar concept. It is a so-called “average radius” where all of the mass of an object is concentrated. For instance, the radius of gyration of a flywheel is r/√2. This means that if the entire mass of the flywheel was represented as a thin hoop, then the radius of that hoop would be r/√2, i.e. the radius would be smaller than the radius of the flywheel. The moment of inertia of any object can be written in terms of its radius of gyration, or

$$I = m \times k^2$$

where k is the radius of gyration.

Second Moment of Area

The second moment of area, or moment of inertia, of an area element dA is defined as

$$I_X = \int_A y^2\,dA$$

about the X-axis

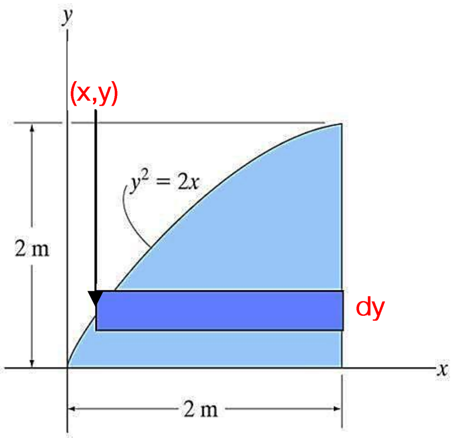
$$I_Y = \int_A x^2\,dA$$

about the Y-axis

Example Pennsylvania State University (2010), viewed 3 Dec.2012, http//www.engr.bd.psu.edu/rxm61/MCHT111/Chapter%2010

Calculate the moment of inertia of the shaded area (light blue)

(a) about the X-axis.



Solution

Use a horizontal element (dark blue): dA = (2-x) × dy

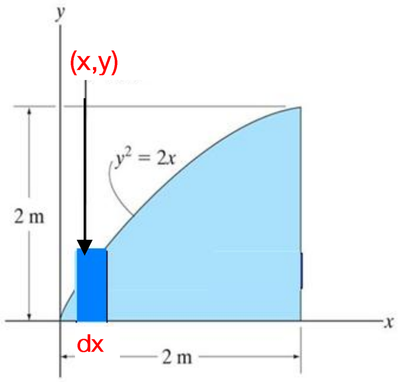
$$dA = (2-x)dy = \left(2-\frac{y^2}{2}\right)dy \qquad \text{since} \quad x = y^2/2$$

$$I_x = \int y^2 dA = \int_0^2 y^2\left(2-\frac{y^2}{2}\right)dy$$

$$I_x = \frac{2}{3}y^3 - \frac{1}{10}y^5\Big|_0^2 = 2.13\ \text{m}^4$$

Note: A vertical element could be used above, however the calculations are harder.

(b) about the Y-axis.



Use a vertical element (dark blue): dA = y × dx

$$I_y = \int x^2 dA = \int_0^2 x^2 y dx \qquad \text{since} \quad y = \sqrt{2x}$$

$$I_y = \int_0^2 x^2\left(\sqrt{2x}\right)dx = \sqrt{2}\int_0^2 x^{2.5}dx$$

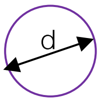
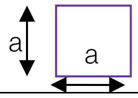
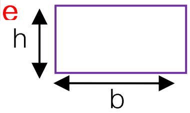
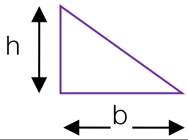
$$I_y = \frac{\sqrt{2}}{3.5}x^{2.5}\Big|_0^2 = 4.57\ \text{m}^4$$

Note: A horizontal element could be used above, however the calculations are harder.

Second Moment of Area: Simple and Composite Shapes

(Ivanoff, V, 2010, *Engineering Mechanics*, McGraw-Hill)

Moments of inertia of simple geometrical shapes are shown below.

Shape	Area A	Position of Centroid	Centroidal moment of inertia I_C
Circle 	$\frac{\pi d^2}{4}$	at centre	$\frac{\pi d^4}{64}$
Square 	a^2	at intersection of diagonals	$\frac{a^4}{12}$
Rectangle 	bh	at intersection of diagonals	$\frac{bh^3}{12}$
Triangle 	$\frac{bh}{2}$	at intersection of medians (⅓ of altitude)	$\frac{bh^3}{36}$

Parallel Axis Theorem

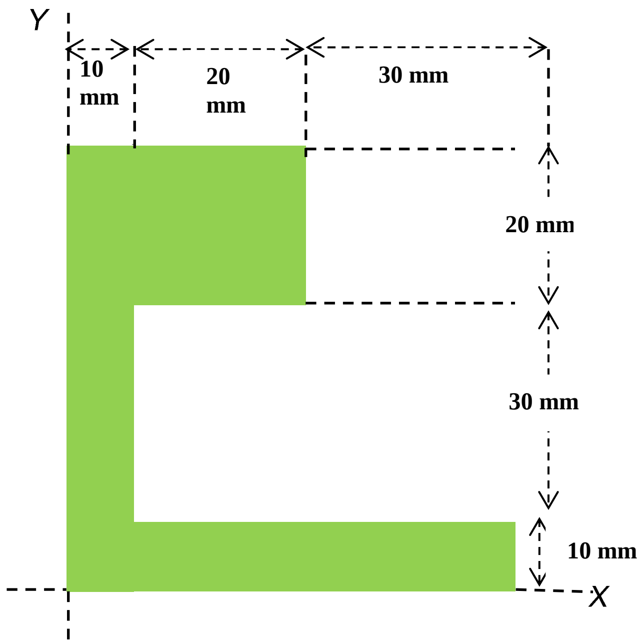
The moment of inertia about any other axis parallel to the centroidal axis is given by

$$I = I_C + Ad^2$$

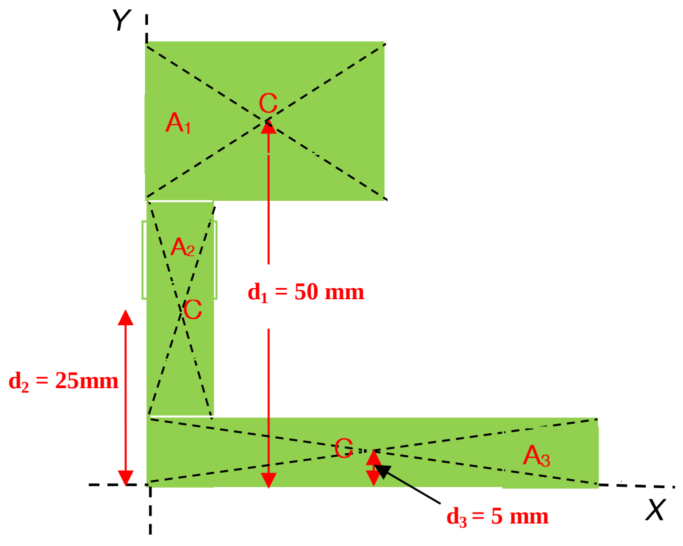
where I is the moment of inertia about any axis; A is the area; d is the distance between the axes; I_C is the moment of inertia about the centroidal axis

Example   Moment of inertia of a composite shape

Calculate the moment of inertia of the shaded area (below) about the X-axis



Solution



Note:

- Divide area into 3 rectangular elements A_1, A_2, and A_3.
- The centroids C of each are shown by the intersection of the diagonals for each rectangular element.
- d_1, d_2, d_3 are the distances from the centroid of each element to the X-axis.

Element	Area A	Distance d of X-axis from centroid	Centroidal moment of inertia $I_C = \frac{bh^3}{12}$	Transfer term Ad^2	Transferred moment of inertia $I = I_C + Ad^2$
1	30×20 =600	50	$\frac{30 \times 20^3}{12} = 20000$	600×50^2 =1500000	20000 + 1500000 = 1520000
2	30×10 =300	25	$\frac{10 \times 30^3}{12} = 22500$	300×25^2 =187500	22500 + 187500 = 210000
3	60×10 =600	5	$\frac{60 \times 10^3}{12} = 5000$	600×5^2 =15000	5000 + 15000 = 20000
				Σ I =	1 750 000 mm^4

Answer: The total moment of inertia of the area about the X-axis is $\Sigma I = 1\,750\,000\ mm^4$