RMIT UNIVERSITY

M9. INVERSE OF A 3X3 MATRIX

Definition

If A is a square matrix and B is another square matrix of the same size, such that

$$AB = BA = I$$

then we call B the inverse of A. The inverse of A is written as A^{-1}.

Therefore

$$AA^{-1} = A^{-1}A = I$$

Not every square matrix has an inverse.

If the determinant of a matrix equals zero the inverse does not exist and the matrix is called **singular**.

If the determinant is not equal to zero then inverse exists and the matrix is called **non-singular or invertible.**

Note: $A^{-1} \neq \frac{1}{A}$. Matrix division is not possible.

Finding the inverse of a matrix

If $AB = I$ then, by multiplying both sides by A^{-1} (assuming A^{-1} exists)

$$A^{-1}AB = A^{-1}I$$
$$\therefore IB = A^{-1}$$
$$\therefore B = A^{-1}$$

Method.

Use the matrix equation $AB = I$ and solve for B.

Form the augmented matrix $[A \vdots I]$ and use row operations to convert this to $[I \vdots B]$

The matrix B is then A^{-1}, the inverse of A.

Inverse of a 3x3 matrix.

Example 1

Find the inverse of $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 2 \\ 1 & 1 & 2 \end{bmatrix}$

The augmented matrix is $\left[\begin{array}{ccc:ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ -1 & 2 & 2 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$

Use row operations to convert the first three columns of the augmented matrix to the 3x3 identity matrix. i.e, 1's the main diagonal and zeros elsewhere in the first three columns.

In each step the numbers being converted to zero are shown in **bold.**

Use row one to get zeros below the '1' in the first column. Adding row one to row two will give zero instead of '-1' and subtracting row one from row three will give zero instead of '1'.

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ \mathbf{-1} & 2 & 2 & 0 & 1 & 0 \\ \mathbf{1} & 1 & 2 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ R_2 + R_1 \to R_2 \\ R_3 - R_1 \to R_3 \end{array} \left[\begin{array}{ccc:ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right]$$

To get a '1' in the second column of the main diagonal interchange row two and row three.

$$\begin{bmatrix} 1 & 0 & 1 & \vdots & 1 & 0 & 0 \\ 0 & \mathbf{2} & 3 & \vdots & 1 & 1 & 0 \\ 0 & 1 & 1 & \vdots & -1 & 0 & 1 \end{bmatrix} \quad R_2 \leftrightarrow R_3 \quad \begin{bmatrix} 1 & 0 & 1 & \vdots & 1 & 0 & 0 \\ 0 & 1 & 1 & \vdots & -1 & 0 & 1 \\ 0 & 2 & 3 & \vdots & 1 & 1 & 0 \end{bmatrix}$$

To get a zero instead of the '2' the second column, subtract two times row two from row three.
As a bonus this also gives a '1' in row three of column three.

$$\begin{bmatrix} 1 & 0 & 1 & \vdots & 1 & 0 & 0 \\ 0 & 1 & 1 & \vdots & -1 & 0 & 1 \\ 0 & \mathbf{2} & 3 & \vdots & 1 & 1 & 0 \end{bmatrix} R_3 - 2R_2 \rightarrow R_3 \begin{bmatrix} 1 & 0 & 1 & \vdots & 1 & 0 & 0 \\ 0 & 1 & 1 & \vdots & -1 & 0 & 1 \\ 0 & 0 & 1 & \vdots & 3 & 1 & -2 \end{bmatrix}$$

Use row three to get zeros above the '1' in row three column three.
Subtract row three from row one and also from row three to get zeros in the respective rows.

$$\begin{bmatrix} 1 & 0 & \mathbf{1} & \vdots & 1 & 0 & 0 \\ 0 & 1 & \mathbf{1} & \vdots & -1 & 0 & 1 \\ 0 & 0 & 1 & \vdots & 3 & 1 & -2 \end{bmatrix} \begin{matrix} R_1 - R_3 \rightarrow R_1 \\ R_2 - R_3 \rightarrow R_2 \end{matrix} \begin{bmatrix} 1 & 0 & 0 & \vdots & -2 & -1 & 2 \\ 0 & 1 & 0 & \vdots & -4 & -1 & 3 \\ 0 & 0 & 1 & \vdots & 3 & 1 & -2 \end{bmatrix}$$

The first three columns have become the identity matrix and the last three columns are the inverse matrix, A^{-1}.

Therefore $A^{-1} = \begin{bmatrix} -2 & -1 & 2 \\ -4 & -1 & 3 \\ 3 & 1 & -2 \end{bmatrix}$ Check by confirming $A^{-1}A = I$

Example 2

Find the inverse of $B = \begin{bmatrix} 1 & -1 & -1 \\ 3 & -2 & 1 \\ -1 & 1 & 3 \end{bmatrix}$

The augmented matrix is $\begin{bmatrix} 1 & -1 & -1 & \vdots & 1 & 0 & 0 \\ 3 & -2 & 1 & \vdots & 0 & 1 & 0 \\ -1 & 1 & 3 & \vdots & 0 & 0 & 1 \end{bmatrix}$

The working is shown below.

$$\begin{bmatrix} 1 & -1 & -1 & \vdots & 1 & 0 & 0 \\ 3 & -2 & 1 & \vdots & 0 & 1 & 0 \\ -1 & 1 & 3 & \vdots & 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_2 - 3R_1 \rightarrow R_2 \\ R_3 + R_1 \rightarrow R_3 \end{matrix} \begin{bmatrix} 1 & 1 & -1 & \vdots & 1 & 0 & 0 \\ 0 & 1 & 4 & \vdots & -3 & 1 & 0 \\ 0 & 0 & 2 & \vdots & 1 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & -1 & \vdots & 1 & 0 & 0 \\ 0 & 1 & 4 & \vdots & -3 & 1 & 0 \\ 0 & 0 & 2 & \vdots & 1 & 0 & 1 \end{bmatrix} {}^{R_3}/_2 \rightarrow R_3 \begin{bmatrix} 1 & -1 & -1 & \vdots & 1 & 0 & 0 \\ 0 & 1 & 4 & \vdots & -3 & 1 & 0 \\ 0 & 0 & 1 & \vdots & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & -1 & \vdots & 1 & 0 & 0 \\ 0 & 1 & 4 & \vdots & -3 & 1 & 0 \\ 0 & 0 & 1 & \vdots & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{matrix} R_1 + R_3 \rightarrow R_1 \\ R_2 - 4R_3 \rightarrow R_2 \end{matrix} \begin{bmatrix} 1 & -1 & 0 & \vdots & \frac{3}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 & \vdots & -5 & 1 & -2 \\ 0 & 0 & 1 & \vdots & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 & \vdots & \frac{3}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 & \vdots & -5 & 1 & -2 \\ 0 & 0 & 1 & \vdots & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} R_1 + R_2 \rightarrow R_1 \begin{bmatrix} 1 & 0 & 0 & \vdots & \frac{-7}{2} & 1 & \frac{-3}{2} \\ 0 & 1 & 0 & \vdots & -5 & 1 & -2 \\ 0 & 0 & 1 & \vdots & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

Therefore $B^{-1} = \begin{bmatrix} \frac{-7}{2} & 1 & \frac{-3}{2} \\ -5 & 1 & -2 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -7 & 2 & -3 \\ -10 & 2 & -4 \\ 1 & 0 & 1 \end{bmatrix}$

Exercises

Find the inverse (if it exists) of:

1. $\begin{bmatrix} 0 & 0 & -1 \\ 2 & 1 & 2 \\ 5 & 3 & 0 \end{bmatrix}$ **2.** $\begin{bmatrix} 1 & 2 & 0 \\ -1 & 3 & 1 \\ 2 & 5 & 2 \end{bmatrix}$ **3.** $\begin{bmatrix} 1 & 2 & 0 \\ 1 & 2 & 1 \\ 1 & 2 & 4 \end{bmatrix}$ **4.** $\begin{bmatrix} -2 & 2 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -2 \end{bmatrix}$

Answers

1. $\begin{bmatrix} 6 & 3 & -1 \\ -10 & -5 & 2 \\ -1 & 0 & 0 \end{bmatrix}$ **2.** $\frac{1}{9}\begin{bmatrix} 1 & -4 & 2 \\ 4 & 2 & -1 \\ -11 & -1 & 5 \end{bmatrix}$ **3.** No inverse. **4.** $\frac{1}{4}\begin{bmatrix} 1 & 4 & 2 \\ 3 & 4 & 2 \\ -1 & 0 & -2 \end{bmatrix}$