M7 Types of Solutions

RMIT UNIVERSITY

$$x + 2y - z = -3$$
$$2x - 3y + 2z = 13$$
$$-x + 5y - 4z = -19$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & -3 & 2 \\ -1 & 5 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 13 \\ -19 \end{bmatrix}$$

For any system of equations, there may be:

1. Infinitely many solutions
2. No solution
3. A unique solution

This module discusses these possibilities.

Coefficient Matrix

Consider the following system of equations:

$$\begin{aligned} 2x + 4y - z &= 9 \\ x - y + 2z &= -4 \\ -x + y - z &= 3 \end{aligned}$$

These may be written in the matrix form:

$$\begin{bmatrix} 2 & 4 & -1 \\ 1 & -1 & 2 \\ -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ -4 \\ 3 \end{bmatrix}.$$

The matrix

$$\begin{bmatrix} 2 & 4 & -1 \\ 1 & -1 & 2 \\ -1 & 1 & -1 \end{bmatrix}$$

is called the coefficient matrix because it contains the coefficients of the variables x, y and z in the system of equations.

What Does the Coefficient Matrix Tell Us?

The coefficient matrix can be used to find information about the number of solutions to a system of equations. For any system of equations, if the coefficient matrix is:

1. Singular (that is the determinant $=0$) there will be an infinite number of solutions or no solutions.
2. Non-singular (determinant $\neq 0$) there will be a unique solution.

Infinitely Many Solutions

Example 1

Consider the system of equations

$$\begin{aligned} x+2y-z&=3\\ x+3y&=4\\ -x-y+2z&=-2 \end{aligned}$$

The augmented matrix for this system is

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3\\ 1 & 3 & 0 & 4\\ -1 & -1 & 2 & -2 \end{array}\right].$$

Using elementary row operations we get:

$$\begin{array}{r} \\ R_2' = R_2 - R_1\\ R_3' = R_3 + R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3\\ 0 & 1 & 1 & 1\\ 0 & 1 & 1 & 1 \end{array}\right]$$

$$\begin{array}{r} \\ \\ R_3' = R_3 - R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3\\ 0 & 1 & 1 & 1\\ 0 & 0 & 0 & 0 \end{array}\right]$$

Row 3 tells us that $0x+0y+0z=0$. This is true for any value of x, y or z. So we are able to set any one[1] of these variables to some number $t \in \mathbb{R}$. For this example[2], we set $z=t$. From R_2 and substituting t for z we have

$$\begin{aligned} y+z&=1\\ y+t&=1\\ y&=1-t. \end{aligned}$$

Now from R_1, substituting $z=t$, $y=1-t$ we have

$$\begin{aligned} x+2y-z&=3\\ x+2(1-t)-t&=3\\ x+2-2t-t&=3\\ x&=1+3t. \end{aligned}$$

[1] It can only be one variable as there other equations to be satisfied in R_2 and R_1.

[2] Even though we set $z=t$, we could set either $x=t$ or $y=t$. We still get a solution to the system of equations but it will look different.

So the solution is $x = 1 + 3t$, $y = 1 - t$ and $z = t$ where $t \in \mathbb{R}$.[3]

In general if the reduced augmented matrix has one or more rows of zeros, there will be an infinite number of solutions to the system of equations.[4]

[3] As there are an infinite number of choices for t, there an infinite number of solutions to the system of equations.

[4] You may recall that the determinant of a matrix with a row of zeros is zero. Hence the matrix is singular.

Example 2

Consider the system of equations

$$\begin{aligned} x + 2y - z &= 3 \\ 2x + 4y - 2z &= 6 \\ -5x - 10y + 5z &= -15 \end{aligned}$$

The augmented matrix for this system is

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & 4 & -2 & 6 \\ -5 & -10 & 5 & -15 \end{array}\right].$$

Using elementary row operations we get

$$\begin{array}{l} \\ R_2' = R_2 - 2R_1 \\ R_3' = R_3 + 5R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

We have only one equation and three unknowns. In this case we must let two of the three variables be free. For example, we set $z = t$, where $t \in \mathbb{R}$ and $y = s$, where $s \in \mathbb{R}$.[5] Substituting into R_1 we get

$$\begin{aligned} x + 2y - z &= 3 \\ x + 2s - t &= 3 \\ x &= 3 - 2s + t. \end{aligned}$$

So the solution is $x = 3 - 2s + t$, $y = s$, $z = t$ where $s \in \mathbb{R}$ and $t \in \mathbb{R}$.

[5] It doesn't matter which two variables are selected to be a parameter s or t. We could set $x = s$ and $y = t$. Substituting in row 1 gives:

$$\begin{aligned} s + 2t - z &= 3 \\ z &= s + 2t - 3. \end{aligned}$$

While this looks different to the solution at left, it is still a valid solution to the set of equations.

No Solutions

Consider the system of equations

$$\begin{aligned} x + 2y - 2z &= 3 \\ 2x + y + z &= 4 \\ 3y - 5z &= -1. \end{aligned}$$

The augmented matrix for this system is

$$\left[\begin{array}{ccc|c} 1 & 2 & -2 & 3 \\ 2 & 1 & 1 & 4 \\ 0 & 3 & -5 & -1 \end{array}\right].$$

Using elementary row operations we get:

$$R_2' = R_2 - 2R_1 \left[\begin{array}{ccc|c} 1 & 2 & -2 & 3 \\ 0 & -3 & 5 & -2 \\ 0 & 3 & -5 & -1 \end{array}\right]$$

$$R_3' = R_3 + R_2 \left[\begin{array}{ccc|c} 1 & 2 & -2 & 3 \\ 0 & -3 & 5 & -1 \\ 0 & 0 & 0 & -3 \end{array}\right].$$

In this case, R_3 says that $0x + 0y + 0z = -3$. This is not possible because regardless of the choice of x, y and z the LHS must be zero. In this case there are no solutions and we call the system of equations inconsistent.

Unique Solutions

Consider the system of equations

$$\begin{aligned} 2x + 4y - z &= 9 \\ x - y + 2z &= -4 \\ -x + y - z &= 3 \end{aligned}$$

The augmented matrix for this system is

$$\left[\begin{array}{ccc|c} 2 & 4 & -1 & 9 \\ 1 & -1 & 2 & -4 \\ -1 & 1 & -1 & 3 \end{array}\right].$$

Using elementary row operations we get

$$\begin{array}{l} R_1' = R_2 \\ R_2' = R_1 \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 2 & -4 \\ 2 & 4 & -1 & 9 \\ -1 & 1 & -1 & 3 \end{array}\right]$$

$$\begin{array}{l} R_2' = R_2 - 2R_1 \\ R_3' = R_3 + R_1 \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 2 & -4 \\ 0 & 6 & -5 & 17 \\ 0 & 0 & 1 & -1 \end{array}\right].$$

From row 3 we have $z = -1$. Substituting this into row 2 we have

$$\begin{aligned} 6y - 5z &= 17 \\ 6y + 5 &= 17 \\ 6y &= 12 \\ y &= 2. \end{aligned}$$

Substituting for z and y in row 1 gives:

$$\begin{aligned} 2x + 4y - z &= 9 \\ 2x + 4\,(2) - (-1) &= 9 \\ 2x + 8 + 1 &= 9 \\ 2x &= 0 \\ x &= 0. \end{aligned}$$

So the solution is $x = 0$, $y = 2$, $z = -1$.

Exercises

Find the solutions to the following systems of equations:

1. $$\begin{aligned} x + 2y - z &= 5 \\ 3x - y + 2z &= 1 \\ 2x - y + 3z &= 0 \end{aligned}$$

2. $$\begin{aligned} x + 2y - 2z &= 7 \\ 2x - y - z &= -3 \\ -x - 2y + 2z &= 1 \end{aligned}$$

3. $$\begin{aligned} 2x - y + 3z &= -3 \\ x + 2y + z &= -4 \\ 4x + 3y + 5z &= -11 \end{aligned}$$

4. $$\begin{aligned} x + 2y - z &= 3 \\ -x + y - 2z &= -3 \\ 3x + 3z &= 9 \end{aligned}$$

Answers

1. $x = 1$, $y = 2$, $z = 0$
2. No solution
3. $x = -\frac{7t}{5} - 2$, $y = \frac{t}{5} - 1$, $z = t$ (Your answer may be different if you choose $y = t$ or $x = t$.)
4. $x = 3 - t$, $y = t$, $z = t$ (Your answer may be different if you choose $y = t$ or $x = t$.)