RMIT UNIVERSITY

M5. SPECIAL MATRICES

Transpose of a matrix
The transpose of a matrix A is denoted A^T and found by interchanging the rows and columns. The first row becomes the first column, the second row becomes the second column etc.

If A is an mxn matrix then A^T is an nxm matrix.

Examples

1. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ then by interchanging rows and columns, $A^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$

In this case A is a 3 x 2 matrix and its transpose, A^T is a 2 x 3 matrix.

2. If $B = \begin{bmatrix} 1 & -2 \\ 7 & 5 \end{bmatrix}$ then $B^T = \begin{bmatrix} 1 & 7 \\ -2 & 5 \end{bmatrix}$

3. If $C = \begin{bmatrix} 5 & 2 \\ 2 & 4 \end{bmatrix}$ then $C^T = \begin{bmatrix} 5 & 2 \\ 2 & 4 \end{bmatrix}$

Note in this case that $C = C^T$. C is called a **symmetric** matrix.

Symmetric matrix.
A symmetric matrix is a square matrix which is equal to its transpose.

> If $A = A^T$ then A is a symmetric matrix.

A symmetric matrix is also symmetric about its leading diagonal (top left to bottom right).

Examples
Matrices D, E and F are symmetric matrices.

$D = \begin{bmatrix} -1 & 3 \\ 3 & 4 \end{bmatrix}$ $E = \begin{bmatrix} 3 & 4 & 5 \\ 4 & -2 & -3 \\ 5 & -3 & 1 \end{bmatrix}$ $F = \begin{bmatrix} 1 & -2 & 4 \\ -2 & 3 & -4 \\ 4 & -4 & 2 \end{bmatrix}$

and $D^T = \begin{bmatrix} -1 & 3 \\ 3 & 4 \end{bmatrix} = D$ $E^T = \begin{bmatrix} 3 & 4 & 5 \\ 4 & -2 & -3 \\ 5 & -3 & 1 \end{bmatrix} = E$ $F^T = \begin{bmatrix} 1 & -2 & 4 \\ -2 & 3 & -4 \\ 4 & -4 & 2 \end{bmatrix} = F$

Orthogonal matrix.
A square matrix A is orthogonal if $A^T A = AA^T = I$ where I is the unit matrix.

> If $A^T A = AA^T = I$ then A is an orthogonal matrix

Because $A^{-1}A = AA^{-1} = I$ this implies that, **for an orthogonal matrix, $\boldsymbol{A^T = A^{-1}}$**
This can be useful for finding the inverse of an orthogonal matrix as it is often much easier to find the transpose than the inverse of a matrix.

Notes:

- The determinant of an orthogonal matrix is either $+1$(a rotation matrix) or -1(a reflection matrix)
- The rows of an orthogonal matrix are mutually orthogonal (perpendicular) unit vectors
- The columns of an orthogonal matrix are mutually orthogonal unit vectors

Examples

Rotation matrix

$A = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$ is an orthogonal matrix (rotates points, lines and regions through an angle of θ° anticlockwise.)

$$|A| = \begin{vmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{vmatrix} = cos^2\theta + sin^2\theta = 1$$

$$A^T = \begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix} = A$$

$$A^T A = \begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix}\begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad AA^T = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}\begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

That is $A^T A = AA^T = I$

$$A^{-1} = \frac{1}{\begin{vmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{vmatrix}}\begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix} = \begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix}$$

That is $A^T = A^{-1}$

Reflection matrix

$B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ is an orthogonal matrix (a reflection around the *x*-axis)

$|B| = -1$ and $B^T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

$B^T B = BB^T = I$ and $B^T = B^{-1}$

Exercise 1
Given:

$$A = \begin{bmatrix} 3 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -3 \\ -3 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & 0 & 5 \\ 0 & -2 & 2 \\ 5 & 2 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 2 & -1 & 5 \\ 5 & -1 & 2 \end{bmatrix} \qquad E = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

a) Write down the transpose of each of the above matrices.
b) Which of the above matrices are symmetric?

Exercise 2
Given:

$$L = \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix} \quad M = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} & 0 \\ \frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad N = \frac{1}{3}\begin{bmatrix} 2 & -2 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & -2 \end{bmatrix} \quad P = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \quad Q = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

a) Which of the matrices L, M, N, P, Q are orthogonal?
b) If the matrix is orthogonal, write down its inverse.
c) Which of the matrices L, M, N, P, Q is a rotation matrix?

Answers
Exercise 1

a) $A^T = \begin{bmatrix} 3 \\ -2 \end{bmatrix} \quad B^T = \begin{bmatrix} 1 & -3 \\ -3 & 0 \end{bmatrix} \quad C^T = \begin{bmatrix} 2 & 0 & 5 \\ 0 & -2 & 2 \\ 5 & 2 & 1 \end{bmatrix} \quad D^T = \begin{bmatrix} 2 & 5 \\ -1 & -1 \\ 5 & 2 \end{bmatrix} \quad E^T = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$

b) B and C

Exercise 2

a) M, N, Q

b) $M^{-1} = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} & 0 \\ \frac{-4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad N^{-1} = \frac{1}{3}\begin{bmatrix} 2 & 1 & 2 \\ -2 & 2 & 1 \\ 1 & 2 & -2 \end{bmatrix} \quad Q^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

c) $M \qquad (|M| = 1)$