LT2.1 LAPLACE TRANSFORMS: ELECTRICAL CIRCUITS

Question
- For an LRC circuit given by

\[ L \frac{di}{dt} + Ri + \frac{1}{C} \int idt = E_0 \quad \frac{dq}{dt} (0) = q(0) = 0 \quad \ldots \ldots (1) \]

and \( L = 1, \quad R = 3, \quad C = 0.5, \quad E_0 = 10 \) find

(a) the charge \( q(t) \) on the capacitor
(b) the resulting current \( i(t) \) in the circuit at time \( t \) using Laplace transforms.

Solution

(a) Since \( i = \frac{dq}{dt} \) becomes

\[ L \frac{d^2q}{dt^2} + R \frac{dq}{dt} + \frac{q}{C} = E_0 \quad \ldots \ldots (2) \]

Note: \( \int i \, dt = \int \frac{dq}{dt} \, dt = \int dq = q \)

Recall:
\[ L[q(t)] = Q(s) \quad ; \quad L[q'(t)] = sQ(s) - q(0) \]
\[ L[a] = \frac{a}{s} \quad ; \quad L[q''(t)] = s^2Q(s) - s q(0) - q'(0) \]

Taking Laplace Transforms of (2) and substituting for \( L, R, C \) & \( E_0 \):

\[ (s^2Q(s) - s q(0) - q'(0)) + 3(5Q(s) - q(0)) + 2Q(s) = \frac{10}{s} \]

Substituting initial values \( \frac{dq}{dt}(0) = q'(0) = 0 \) gives

\[ s^2 Q(s) + 3s Q(s) + 2Q(s) = \frac{10}{s} \]

\( \Rightarrow Q(s) \left[ s^2 + 3s + 2 \right] = \frac{10}{s} \Rightarrow Q(s) = \frac{10}{s(s+1)(s+2)} \)
Resolving into partial fractions gives
\[ 10 = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2} \] ...

Setting \( s = 0 \):
\[ 10 = A \cdot 1 \cdot 2 = 2A \quad \Rightarrow A = 5 \]
\( s = -1 \):
\[ 10 = B \cdot (-1) \cdot 1 = -B \quad \Rightarrow B = -10 \]
\( s = -2 \):
\[ 10 = C \cdot (-2) \cdot (-1) = 2C \quad \Rightarrow C = 5 \]

Substituting back into (3) gives
\[ Q(s) = \frac{5}{s} - \frac{10}{s+1} + \frac{5}{s+2} \]

Taking inverse Laplace transforms:
\[ q(t) = L^{-1} \left[ \frac{5}{s} - \frac{10}{s+1} + \frac{5}{s+2} \right] \]

\[ q(t) = 5 - 10e^{-t} + 5e^{-2t} \]

(b) The resulting current \( i(t) \) at time \( t \) is
\[ i(t) = \frac{dq}{dt} = \frac{d}{dt} \left( 5 - 10e^{-t} + 5e^{-2t} \right) = 10e^{-t} - 10e^{-2t} \]

Note: We could have found the current by taking Laplace transforms of (1), however the mathematics is more complicated.

This question could have been solved more easily using a 2nd order D.E had the question not asked specifically for its solution using Laplace transforms.