Solving Differential Equations Using Laplace Transforms

Example

Given the following first order differential equation, \( \frac{dy}{dx} + y = 3e^{2t} \), where \( y(0)=4 \).

Find \( y(t) \) using Laplace Transforms.

Sol:\nTo begin solving the differential equation we would start by taking the Laplace transform of both sides of the equation.

\[
L\left[ \frac{dy}{dt} + y \right] = L[3e^{2t}]
\]

Taking the Laplace Transform of both sides of the equation.

\[
L\left[ \frac{dy}{dt} \right] + L[y] = 3L[e^{2t}]
\]

Separating terms.

\[
sY - y(0) + Y = 3 \times \frac{1}{s-2}
\]

Transforms as derived from tables.

\[
sY - 4 + Y = \frac{3}{s-2}
\]

Substituting for \( y(0)=4 \)

\[
Y(s + 1) = \frac{3}{s-2} + 4
\]

Taking \( Y \) as a common factor.

\[
Y = \frac{3}{(s-2)(s+1)} + \frac{4}{(s+1)}
\]

Making \( Y \) the subject.

\[
Y = \frac{3}{(s+1)(s-2)} + \frac{4(s-2)}{(s+1)(s-2)}
\]

\[
Y = \frac{4s - 5}{(s-2)(s+1)}
\]

Use partial fractions to expand \( \frac{4s - 5}{(s-2)(s+1)} \)

\[
\therefore \quad \frac{4s - 5}{(s-2)(s+1)} = \frac{A}{s-2} + \frac{B}{s+1}
\]

\[
4s - 5 = A(s + 1) + B(s - 2)
\]

By selecting appropriate values of \( s \), we can solve for \( A & B \).

Letting \( s = -1 \), and substituting into the above equation gives
\[ 4(-1) - 5 = A(-1+1) + B(-1-2) \]
\[ -4 - 5 = A(0) + B(-3) \]
\[ -9 = -3B \]
\[ B = \frac{-9}{-3} = 3 \]

Now let \( s = 2 \), and substitute into the same equation
\[ 4(2) - 5 = A(2+1) + B(2-2) \]
\[ 8 - 5 = A(3) + B(0) \]
\[ 3 = 3A \]
\[ A = \frac{3}{3} = 1 \]

So
\[ \frac{4s - 5}{(s-2)(s+1)} = \frac{1}{s-2} + \frac{3}{s+1} \]

Therefore
\[ Y = \frac{1}{s-2} + \frac{3}{s+1} \]

To obtain a solution \( y(t) \) to the differential equation from \( Y(s) \) we need to find the inverse Laplace transform of \( Y \).
\[ \therefore L^{-1}[Y] = L^{-1} \left[ \frac{1}{s-2} + \frac{3}{s+1} \right] \]

\[ \therefore y(t) = L^{-1} \left[ \frac{1}{s-2} \right] + 3L^{-1} \left[ \frac{1}{s+1} \right] \quad \text{Inverse transforms obtained from tables.} \]

\[ \therefore y(t) = e^{2t} + 3e^{-t} \]
Example

Given the following second order differential equation, \( y'' + y' = 5 \cos 2t \); \( y(0) = 0 \); \( y'(0) = 0 \)

Find \( y(t) \) using Laplace Transforms.

Sol:

\[ L[y''] + L[y'] = L[5 \cos 2t] \]

\[ s^2Y - sy(0) - y'(0) + sY - y(0) = \frac{5s}{s^2 + 4} \]

\[ Y(s^2 + s) = \frac{5s}{s^2 + 4} \]

\[ Y = \frac{5s}{(s^2 + s)(s^2 + 4)} \]

\[ Y = \frac{5s}{(s)(s+1)(s^2 + 4)} = \frac{5}{(s+1)(s^2 + 4)} = \frac{A}{s+1} + \frac{Bs + C}{s^2 + 4} \]

\[ \therefore 5 = A(s^2 + 4) + (Bs + C)(s + 1) \]

An alternative method for solving the unknowns \( A, B, C \) in the above equation is called "Equating coefficients of powers of \( s \):

\[ \begin{align*}
LHS &= RHS \\
\text{s}^0 &: \quad 5 &= 4A + C \quad \text{eq} \, 1.
\text{s}^1 &: \quad 0 &= B + C \quad \text{eq} \, 2.
\text{s}^2 &: \quad 0 &= A + B \quad \text{eq} \, 3.
\end{align*} \]

From eq\,3 \quad \quad \quad A = -B

From eq\,2 \quad \quad \quad C = -B

Substitute in eq\,1 \quad \quad \quad 5 = 4(-B) + (-B)

\[ \therefore A = 1, \quad B = -1, \quad C = 1 \]

\[ \therefore Y = \frac{1}{s+1} + \frac{s+1}{s^2+4} = \frac{1}{s+1} - \frac{s}{s^2+4} + \frac{1}{s^2+4} \]

\[ \therefore y(t) = L^{-1}[Y] = L^{-1}\left[ \frac{1}{s+1} \right] - L^{-1}\left[ \frac{s}{s^2+4} \right] + L^{-1}\left[ \frac{1}{s^2+4} \right] \]

From tables:

\[ y(t) = e^{-t} - \cos 2t + \sin 2t \]
Using partial fractions determine the inverse Laplace Transforms of the following expressions.

a. \( \frac{5s+2}{(s+1)(s+2)} \)  
b. \( \frac{3s+4}{(s+2)(s+3)} \)  
c. \( \frac{4s+1}{(s+3)(s+4)} \)  
d. \( \frac{6s-5}{(s+5)(s+3)} \)  
e. \( \frac{4s+1}{s(s+2)(s+3)} \)  
f. \( \frac{2s-8}{(s+2)(s^2+7s+6)} \)  

Answers

a. \( 8e^{-2t} - 3e^{-t} \)  
b. \( 5e^{-3t} - 2e^{-2t} \)  
c. \( 15e^{-4t} - 11e^{-3t} \)  
d. \( \frac{35}{2}e^{-5t} - \frac{23}{2}e^{-3t} \)  
e. \( \frac{1}{6} + \frac{7}{2}e^{-2t} - \frac{11}{3}e^{-3t} \)  
f. \( 3e^{-2t} - e^{-6t} - 2e^{-t} \)  

Solve the differential equations using Laplace Transform methods.

a. \( \frac{dx}{dt} + x = 0 \); \( x(0) = 3 \)  
b. \( \frac{dx}{dt} + x = 9e^{2t} \); \( x(0) = 3 \)  
c. \( y'' - y = -t^2 \)  
y(0) = 2 ; \( y'(0) = 0 \)  
d. \( y'' - y' - 2y = -2 \)  
y(0) = 2 ; \( y'(0) = 0 \)  
e. \( x'' + 2x' + 2x = e^{-t} \)  
x(0) = x'(0) = 0  
f. \( x'' - 5x' + 6x = 6t - 4 \)  
x(0) = 1 ; \( x'(0) = 2 \)  

Answers

a. \( x(t) = 3e^{-t} \)  
b. \( x(t) = 3e^{2t} \)  
c. \( y(t) = 2 + t^2 \)  
d. \( y(t) = 1 + \frac{2}{3}e^{-t} + \frac{1}{3}e^{2t} \)  
e. \( x(t) = e^{-t} - e^{-t} \cos t \)  
f. \( x(t) = \frac{1}{6} + t + \frac{3}{2}e^{2t} - \frac{2}{3}e^{-3t} \)