Integral Calculus
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Antidifferentiation

Antidifferentiation reverses the process from differentiation. Given a derivative \( f'(x) \) the task is to find the original function \( f(x) \).

If \( f(x) = \left( \frac{x^3}{3} \right) \), then \( f'(x) = x^2 \), therefore \( \frac{x^3}{3} \) is an antiderivative of \( x^2 \).

If \( f(x) = \left( \frac{x^3}{3} + 1 \right) \), then \( f'(x) = x^2 \), therefore \( \frac{x^3}{3} + 1 \) is an antiderivative of \( x^2 \).

If \( f(x) = \left( \frac{x^3}{3} + 2 \right) \), then \( f'(x) = x^2 \), therefore \( \frac{x^3}{3} + 2 \) is an antiderivative of \( x^2 \).

......and so on.

In general:

\[
\frac{dy}{dx} = x^n, \quad y = \frac{x^{n+1}}{n+1} + c, \quad (n \neq -1) \text{ where } c \text{ is a constant}
\]

This is sometimes referred to as the 'power rule' and applies for positive, negative and fractional values of \( n \) except \( n = -1 \)

Examples

1. Given \( \frac{dy}{dx} = 2x \) find the antiderivative

\[
y = 2 \frac{x^{1+1}}{1+1} + c \quad \text{[add one to the power of } x, \text{ divide by the new power, add a constant]} \\
= 2 \frac{x^2}{2} + c \\
= x^2 + c
\]

2. Given \( \frac{dy}{dx} = 3, \) find the antiderivative

\[
y = 3 \frac{x^{0+1}}{0+1} + c \quad \text{[add one to the power of } x, \text{ divide by the new power, add a constant]} \\
= 3x + c
\]

3. Given \( \frac{dy}{dx} = x^{-3} \), find the antiderivative

\[
y = \frac{x^{-3+1}}{-3+1} + c \quad \text{[add one to the power of } x, \text{ divide by the new power, add a constant]} \\
= \frac{x^{-2}}{-2} + c \\
= \frac{1}{-2x^2} + c
\]
4. Given \( f'(x) = x^3 \), find an antiderivative

\[
y = \frac{1}{3+1} x^{3+1} + c \quad \text{[add one to the power of } x, \text{ divide by the new power, add a constant]}
\]
\[
y = \frac{1}{4} x^4 + c
\]
\[
y = \frac{2x^3}{3} + c
\]

Exercise
Find an antiderivative of

1) \( x^3 \)  
2) \( s^0 \)  
3) \( \sqrt[3]{x} \)  
4) \( x^{-5} \)  
5) \( 6 \)  
6) \( m^2 \)  
7) \( p^{-\frac{1}{2}} \)

Answers

1) \( \frac{x^4}{4} \)  
2) \( \frac{s^9}{9} \)  
3) \( \frac{3x^4}{4} \)  
4) \( \frac{x^{-4}}{4} \)  
5) \( 6x \)  
(f) \( -m^{-1} = -\frac{1}{m} \)  
(g) \( 2p^{\frac{1}{2}} \)

NB: Any constant added to any of the above also provides a correct answer
THE INDEFINITE INTEGRAL

\[ \int f(x) \, dx \] is read as "the integral of \( f(x) \)" and indicates that we wish to find the antiderivative of \( f(x) \). The process of finding the antiderivative is called integration.

When 'c' is unspecified the result of integration is an indefinite integral.

Operational rules

\[ \int f(x) \pm g(x) \, dx = \int f(x) \, dx \pm \int g(x) \, dx \]

\[ \int k f(x) \, dx = k \int f(x) \, dx \]

Integrating powers of \( x \)

Using the rule for finding the antiderivative of \( x^n \)

\[ \int x^n \, dx = \frac{x^{n+1}}{n+1} + c, \text{ (providing } n \neq -1) \]

Examples

1. \[ \int (x^3 + 4) \, dx = \int x^3 \, dx + \int 4 \, dx \]
   \[ = \frac{x^4}{4} + 4x + c \text{ [Each term can be integrated separately]}
   \]
   \[ = \frac{x^4}{4} + 4x + c \text{ [Only one constant of integration is needed]} \]

2. \[ \int \frac{1}{3\sqrt{x}} \, dx = \frac{1}{3} \int x^{-\frac{1}{2}} \, dx \]
   \[ = \frac{1}{3} \cdot \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c \]
   \[ = \frac{2}{3} x^{\frac{1}{2}} + c \text{ [Using } \int k f(x) \, dx = k \int f(x) \, dx \]

3. \[ \int \frac{3s^4 - s^3 + 7}{s^2} \, ds = \int (3s^2 - s + 7s^{-2}) \, ds \]
   \[ = \frac{3s^3}{3} - \frac{s^2}{2} - \frac{7}{s} + c \text{ [it is conventional to give the answer in the same form as the question]} \]

[Divide each term by \( s^2 \)]

Integrating powers of linear functions of \( x \)

Functions of the form \( f(x) = (ax + b)^n \) can be integrated using the following rule:

\[ \int (ax + b)^n \, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c, \text{ (providing } n \neq -1) \]

Examples

1. \[ \int (8x - 5)^9 \, dx = \frac{(8x-5)^{9+1}}{8(9+1)} + c \]
   \[ = \frac{(8x-5)^{10}}{80} + c \]

2. \[ \int (7 - 2x)^{-3} \, dx = \frac{(7-2x)^{-3+1}}{-2(-3+1)} + c \]
   \[ = \frac{(7-2x)^{-2}}{6} + c \]
   \[ = \frac{1}{6(7-2x)^2} + c \]
Exercises

1. Find the following integrals.

(a) $\int 3x^2 \, dx$  
(b) $\int 4x^5 - 2x^3 + 9 \, dx$  
(c) $\int x^2 + \frac{1}{x^2} \, dx$

(d) $\int \frac{t^3 - 2t^3 + 1}{2t^2} \, dt$  
(e) $\int s + \frac{2}{3\sqrt{s^3}} \, ds$

2. (a) $\int (5x + 1)^4 \, dx$  
(b) $\int \sqrt{x - 9} \, dx$  
(c) $\int \frac{2}{(x - 3)^{3/2}} \, dx$

Answers

1. (a) $x^3 + c$  
(b) $\frac{2x^6}{3} - \frac{x^4}{2} + 9x + c$  
(c) $\frac{x^3}{3} - \frac{1}{x} + c$

(d) $\frac{t^3}{6} - \frac{t^2}{2} + \frac{1}{2t} + c$  
(e) $\frac{x^2}{2} - \frac{4}{3\sqrt{x^3}} + c$

2. (a) $\frac{(5x+1)^5}{25} + c$  
(b) $\frac{2(x-9)^{3/2}}{3} + c$  
(c) $-\frac{1}{(x+3)^2} + c$
INTEGRATION OF FUNCTIONS

Integrals of the form $\frac{1}{x}$
The rule for integrating $x^n$ cannot be used when $n = -1$. (Why?)

Differentiating $\log_e x$ gives $\frac{1}{x}$ therefore an antiderivative of $\frac{1}{x}$ is $\log_e x$

\[
\int \frac{1}{x} \, dx = \log_e|x| + c \\
\text{or more generally} \\
\int \frac{1}{ax+b} \, dx = \frac{1}{a} \log_e|ax+b| + c
\]

NB:
- we cannot take the log of a negative number
- this rule only applies when the denominator is a linear function

Examples
1. \( \int \frac{5}{x} \, dx = 5 \log_e|x| + c \)
2. \( \int \frac{1}{2x-5} \, dx = \frac{1}{2} \log_e|2x-5| + c \quad [a = 2 \text{ and } b = -5] \)
3. \( \int \frac{1}{1-3x} \, dx = -\frac{1}{3} \log_e|1-3x| + c \quad [a = -3 \text{ and } b = 1] \)

Exponential function
If \( f(x) = e^x \) then \( f'(x) = e^x \), therefore an antiderivative of \( e^x \) is \( e^x \)

\[
\int e^x \, dx = e^x + c \\
\text{or more generally} \\
\int e^{ax+b} \, dx = \frac{1}{a} e^{ax+b} + c
\]

Examples
1. \( \int 2e^x \, dx = 2e^x + c \)
2. \( \int e^{-5x+1} \, dx = -\frac{1}{5} e^{-5x+1} + c \quad [a = 5 \text{ and } b = 1] \)
3. \( \int e^{\frac{x}{3}+4} \, dx = \frac{1}{3} e^{\frac{x}{3}+4} + c = 3e^{\frac{x}{3}+4} + c \quad [a = \frac{1}{3} \text{ and } b = 4] \)

Trigonometric functions
Using the derivatives of the trigonometric functions the following integrals can be written down:

\[
\int \sin(x) \, dx = -\cos(x) + c \quad \text{or} \quad \int \sin(ax+b) \, dx = -\frac{1}{a} \cos(ax+b) + c \\
\int \cos(x) \, dx = \sin(x) + c \quad \text{or} \quad \int \cos(ax+b) \, dx = \frac{1}{a} \sin(ax+b) + c \\
\int \sec^2(x) \, dx = \tan(x) + c \quad \text{or} \quad \int \sec^2(ax+b) \, dx = \frac{1}{a} \tan(ax+b) + c
\]
Examples
1. $\int 5 \cos(x) \, dx = 5 \sin(x) + c$
2. $\int 3 \cos(3 - 2x) \, dx = -\frac{3}{2} \sin(3 - 2x) + c \quad [a = -2 \text{ and } b = 3]$
3. $\int \sec^2\left(\frac{x}{2}\right) \, dx = 2 \tan\left(\frac{x}{2}\right) + c \quad [a = \frac{1}{2} \text{ and } b = 0]$

Exercise
Find the following integrals

1. (a) $\int \frac{1}{x+5} \, dx$ (b) $\int \frac{1}{6-5x} \, dx$ (c) $\int \frac{1}{3x+5} \cdot \frac{1}{x-2} \, dx$
2. (a) $\int e^{3x} \, dx$ (b) $\int e^{2-5x} \, dx$ (c) $\int \frac{9e^{3x-4}+5}{e^{2x}} \, dx \quad [\text{Hint: divide through}]$
3. (a) $\int \sec^2(4x) \, dx$ (b) $\int 2 \cos(1 - x) \, dx$ (c) $\int 2 \sin\left(\frac{5-3x}{4}\right) \, dx$

Answers
1. (a) $\log_e|x + 5| + c$ (b) $-\frac{1}{5} \log_e|6 - 5x| + c$ (c) $\frac{1}{3} \log_e|3x + 5| \cdot \log_e|x - 2| + c$
2. (a) $\frac{e^{3x}}{3} + c$ (b) $\frac{e^{2-5x}}{-5} + c$ (c) $9e^{x-4} \cdot \frac{5}{2e^{2x}} + c$
3. (a) $\frac{1}{4} \tan 4x + c$ (b) $-2 \sin(1 - x) + c$ (c) $\frac{8}{3} \cos\left(\frac{5-3x}{4}\right) + c$
DEFINITIVE INTEGRALS

\( \int_a^b f(x) \, dx \) is called the definite integral from \( x = a \) to \( x = b \) where \( a \) is called the lower limit of integration \( b \) is called the upper limit of integration.

\[
\int_a^b f(x) \, dx = [F(x)]_a^b = F(b) - F(a) \quad \text{where} \quad F(x) = \int f(x) \, dx
\]

This can be calculated only if \( f(x) \) is defined for all \( x \) in the interval \( a \leq x \leq b \)

Examples

1. \( \int_2^4 (x + 1) \, dx = \left[ \frac{x^2}{2} + x \right]_2^4 = \left[ \frac{4^2}{2} + 4 \right] - \left[ \frac{2^2}{2} + 2 \right] = 8 \)

2. \( \int_0^\pi (\cos x + e^{-2x}) \, dx = \left[ \sin x - \frac{e^{-2x}\pi}{2} \right]_0^\pi = \left[ \sin \pi - \frac{e^{-2\pi}}{2} \right] - \left[ \sin 0 - \frac{e^{-2\cdot0}}{2} \right] = \left[ 0 - \frac{1}{2} \right] - \frac{1-e^{-2\pi}}{2} = \frac{1-e^{-2\pi}}{2} \)

3. If the work done (measured in joules) in moving an object from point \( a \) to point \( b \) is given by \( W = \int_a^b (3x^2 + 2) \, dx \) find the work done in moving the object from the point \( x = 0 \) to the point \( x = 3 \).

\( \int_0^3 (3x^2 + 2) \, dx = \left[ \frac{3x^3}{3} + 2x \right]_0^3 = \left[ \frac{3(3)^3}{3} + 2(3) \right] - \left[ \frac{3(0)^3}{3} + 2(0) \right] = (3^3 + 6) - (0) = 33 \) joules

Exercises

1. Evaluate:
   (a) \( \int_0^2 (3x^2 + x + 1) \, dx \)   (b) \( \int_0^\pi (\cos x + \sin 2x) \, dx \)   (c) \( \int_{-2}^4 2e^{-3x} \, dx \)

2. The acceleration of a particle is given by \( a(t) = 2t^2 + 3e^t \) m/s\(^2\). If its initial velocity, \( v(0) \), is 2 m/s find the velocity when \( t = 3 \). \([\text{NB: acceleration } a(t) = v'(t)]\)

Answers

1(a) 12   (b) 0   (c) \( \frac{2e^6}{3} - \frac{2e^{-12}}{3} \)
2. 22.88 m/s
AREA UNDER A CURVE

Calculation of Areas

The definite integral \( \int_a^b f(x) \, dx \) gives a measure of the signed area between the graph of \( f(x) \) and the x-axis between the values of \( x = a \) and \( x = b \).

[Image from education.com]

If the area is above the x-axis, the definite integral is positive.

If the area is below the x-axis, the definite integral is negative.

The actual area is the absolute value of the signed area: \( A = \left| \int_a^b f(x) \, dx \right| \)

Note: If the curve crosses the x-axis, the areas above and below the x-axis must be calculated separately.

Examples

1. Given the function \( f(x) = 4x - x^2 \) find the area between the curve and the x-axis

Sketch the graph of \( f(x) = 4x - x^2 \) to determine the area to be calculated.

\[ 4x - x^2 = 0 \Rightarrow x(4 - x) = 0 \Rightarrow x = 0 \text{ and } x = 4 \]

The limits of integration are 0 and 4 and the area is all above the x-axis.

\[
A = \left| \int_0^4 4x - x^2 \, dx \right|
\]

\[
= \left| \left[ 2x^2 - \frac{x^3}{3} \right]_0^4 \right|
\]

\[
= \left| \left( 4(4^2) - \frac{4^3}{3} \right) - \left( 2(0^2) - \frac{0^3}{3} \right) \right|
\]

\[
= \frac{32}{3} \text{ square units}
\]
2. Given the function \( f(x) = x^2 - 1 \) find the area bounded by the curve, the x-axis and the lines \( x = 0 \) and \( x = 2 \).

Sketch the graph of \( f(x) = x^2 - 1 \) to determine the area to be calculated.

Because the function crosses the x-axis in the interval \((0, 2)\) each area must be calculated separately.

\[
A = \left| \int_{0}^{1} x^2 - 1 \, dx \right| + \left| \int_{1}^{2} x^2 - 1 \, dx \right|
\]

\[
= \left| \left[ \frac{x^3}{3} - x \right]_{0}^{1} + \left[ \frac{x^3}{3} - x \right]_{1}^{2} \right|
\]

\[
= \left| \left( \frac{1}{3} - 1 \right) - \left( \frac{0}{3} - 0 \right) \right| + \left| \left( \frac{2}{3} - 2 \right) - \left( \frac{1}{3} - 1 \right) \right|
\]

\[
= \left| -\frac{2}{3} + \frac{4}{3} \right|
\]

\[
= \frac{2}{3} \text{ square units}
\]

**Exercise**

1. Find the area enclosed by the curve \( y = 6x - x^2 \) and the x axis.

2. Find the area bounded by the curve \( y = \sin(x) \), the x-axis and the line \( x = \frac{\pi}{2} \).

3. Find the area bounded by the curve \( y = x^2 - 3x + 2 \) and the x-axis between \( x = 0 \) and \( x = 2 \).

**Answers**

1. 36 square units  
2. 1 square unit  
3. 1 square unit