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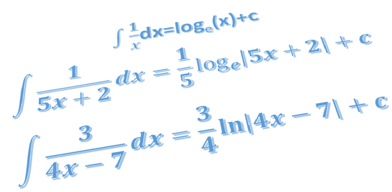

Integration of Functions of the Form $f(x) = \frac{m}{ax+b}$

This module looks at integrals such as $\int \frac{1}{x}dx$ and $\int \frac{1}{3x-1}dx$.

The Power Rule for Integration

One of the most important rules for integration is the power rule which says:[1]

> Provided $n \neq -1$ then
> $$\int x^n dx = \frac{1}{n+1}x^{n+1} + c$$
> where c is a constant.

For example:

$$\int x dx = \frac{1}{2}x^2 + c, \quad c \in \mathbb{R}$$
$$\int x^2 dx = \frac{1}{3}x^3 + c, \quad c \in \mathbb{R}$$

The question is, " How do we deal with the case when $n = -1$. That is, how do we integrate $\int (1/x)\, dx$? The answer is we define a new function as follows:

$$\int \frac{1}{x}dx = \log_e |x| + c \quad \text{where c is a constant.}$$

This function, $\log_e |x|$, is called the "natural logarithm". It is sometimes written as $\ln|x|$. Note the use of the absolute value sign. The natural logarithm is not defined for negative arguments.

We can now integrate $f(x) = m/x$ where m is any constant:

$$\int \frac{m}{x}dx = m\int \frac{1}{x}dx$$
$$= m \log_e |x| + c$$

[1] Note that if $n = -1$, $1/(n+1)$ would be $1/0$ which has no meaning. n can be any number (other than -1) - positive, negative or a fraction.

Instead of saying c is a constant, we sometimes write $c \in \mathbb{R}$ which means c is a real number.

For example, the integral of $f(x) = \frac{5}{x}$ with respect to x is $\int \frac{5}{x} dx = 5 \log_e |x| + c \quad c \in \mathbb{R}$. There is a more general rule which covers more cases. It is:

Let a , b and m be constants with $a \neq 0$, then

$$\int \frac{m}{ax+b} dx = \frac{m}{a} \log_e |ax+b| + c$$

where c is a constant.

Examples

1. Integrate $f(x) = \frac{1}{2x}$ with respect to x. [2]
   Solution: Using the rule
   $$\int \frac{1}{2x} dx = \frac{1}{2} \log_e |2x| + c, \quad c \in \mathbb{R}.$$

[2] Here, $m = 1$, $a = 2$ and $b = 0$.

2. Calculate the integral $\int \frac{2}{3x} dx$ [3]
   Solution: Using the rule
   $$\int \frac{2}{3x} dx = \frac{2}{3} \log_e |3x| + c, \quad c \in \mathbb{R}.$$

[3] Here, $m = 2$, $a = 3$ and $b = 0$.

3. Integrate $f(x) = \frac{13}{5x-7}$ with respect to x. [4]
   Solution: Using the rule
   $$\int \frac{13}{5x-7} dx = \frac{13}{5} \ln |5x-7| + c, \quad c \in \mathbb{R}.$$

[4] Here, $m = 13$, $a = 5$ and $b = -7$.

4. Integrate $f(x) = \frac{6}{2-3x}$ with respect to x. [5]
   Solution: Using the rule
   $$\begin{aligned} \int \frac{6}{2-3x} dx &= \frac{6}{-3} \log_e |2-3x| + c, \quad c \in \mathbb{R} \\ &= -2 \log_e |2-3x| + c. \end{aligned}$$

[5] Here, $m = 6$, $a = -3$ and $b = 2$.

5. Integrate $f(x) = \frac{6}{2-3x} - \frac{2}{x+1}$ with respect to x. [6]
   Solution: Using the rule
   $$\begin{aligned} \int \left(\frac{6}{2-3x} - \frac{2}{x+1} \right) dx &= \frac{6}{-3} \log_e |2-3x| - 2 \log_e |x+1| + c, \quad c \in \mathbb{R} \\ &= -2 \log_e |2-3x| - 2 \log_e |x+1| + c \end{aligned}$$

[6] Here, $m = 6$, $a = -3$ and $b = 2$ in the first term on the right hand side and $m = -2$, $a = 1$ and $b = 1$in the second.

Exercises

1. Calculate the following integrals

   $a)\ \int \frac{1}{5x} dx \qquad b)\ \int \frac{3}{2x} dx \qquad c)\ \int \frac{3}{3x-5} dx \qquad d)\ \int \frac{6}{2-7x} dx$

2. Integrate the following functions with respect to x:

$a)\quad f(x) = x^2 - 2/x \qquad b)\quad f(x) = \frac{1}{3x-2} + \frac{3}{1-x} \qquad c)\quad f(x) = \frac{3}{3x-7} - x^2 \qquad d)\quad f(x) = \frac{2}{5x-4} - \frac{3}{2-5x}$

Answers

1a)	$\frac{1}{5}\log_e\lvert x\rvert + c$	1b)	$\frac{3}{2}\log_e\lvert 2x\rvert + c$	1c)	$\frac{3}{2}\log_e\lvert 2x-7\rvert + c$
1d)	$-\frac{6}{7}\log_e\lvert 2-7x\rvert + c$				

2a)	$\frac{1}{3}x^3 - 2\log_e\lvert x\rvert + c$	2b)	$\frac{1}{3}\log_e\lvert 3x-2\rvert - 3\log_e\lvert 1-x\rvert + c$	2c)	$\log\lvert 3x-5\rvert + c$
2d)	$\frac{2}{5}\log_e\lvert 5x-4\rvert + \frac{3}{2}\log_e\lvert 2-5x\rvert + c$				

The Definite Integral

Answers to all the integrals above were functions of x and involved a constant. Such integrals are called indefinite integrals or antiderivatives. They do not have a specific value.

For a function $f(x)$ suppose we can write an antiderivative $F(x) = \int f(x)dx$. We then define a definite integral of a function $f(x)$ from a to b with respect to x using the notation:[7]

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$

where we assume $f(x)$ is defined for all x in the interval $[a, b]$. We call a the lower limit of integration and b the upper limit of integration. The notation $[F(x)]_a^b$ means substitute $x = b$ and $x = a$ into $F(x)$ and then subtract the values. This gets rid of the constant of integration c.

For example, suppose we want to find, $\int_1^3 xdx$. Then [8]

$$\begin{aligned} F(x) &= \int xdx \\ &= \left[\frac{1}{2}x^2 + c\right]_{x=1}^{x=3}, \quad c \in \mathbb{R} \\ &= \left(\frac{1}{2}3^2 + c\right) - \left(\frac{1}{2}1^2 + c\right) \\ &= \left(\frac{9}{2} + c\right) - \left(\frac{1}{2} + c\right) \\ &= \frac{8}{2} \\ &= 4. \end{aligned}$$

[7] This is known as the first fundamental theorem of calculus.

[8] First we find an antiderivative or indefinite integral $\int xdx + c$ and then evaluate this at the limits of integration.

Note that the indefinite integral has a numerical value, in this case 4, and there is no constant c.

Examples

1. Find the integral $\int_1^2 \frac{1}{2x} dx$. [9]
   Solution:
$$\begin{aligned}\int_1^2 \frac{1}{2x} dx &= \left[\frac{1}{2}\log_e|2x| + c\right]_{x=1}^{x=2} \\ &= \frac{1}{2}\log_e|2(2)| + c - \left(\frac{1}{2}\log_e|2\,(1)| + c\right) \\ &= \frac{1}{2}\log_e|4| - \frac{1}{2}\log_e|2| \\ &= \frac{1}{2}\log_e\left|\frac{4}{2}\right| \\ &= \frac{1}{2}\log_e 2.\end{aligned}$$
   Note that it is not usual to include the constant c as we did in lines 1 and 2 of the solution above. The next example shows this.

[9] In this case the antiderivative is $F(x) = \frac{1}{2}\log_e|2x| + c, \quad c \in \mathbb{R}$. Also remember the log laws:
$$\log b - \log a = \log\frac{b}{a}$$
$$\log b + \log a = \log(ba)\,.$$

2. Find the integral $\int_2^3 \frac{13}{5x-7} dx$. [10]
   Solution:
$$\begin{aligned}\int_2^3 \frac{13}{5x-7} dx &= \left[\frac{13}{5}\log_e|5x-7|\right]_2^3 \\ &= \frac{13}{5}\log_e|15-7| - \left(\frac{13}{5}\log_e|10-7|\right) \\ &= \frac{13}{5}\log_e|8| - \left(\frac{13}{5}\log_e|3|\right) \\ &= \frac{13}{5}\log_e\frac{8}{3}.\end{aligned}$$

[10] In this case the antiderivative is $F(x) = \frac{13}{5}\log_e|5x-7| + c, \quad c \in \mathbb{R}$.

Exercises:

1. Find the integral $\int_1^5 \frac{2}{3x} dx$.
2. Calculate the integral $\int_1^2 \frac{2}{3x-1} dx$
3. Find the integral $\int_1^3 \left(x^2 - 2/x\right) dx$. Hint: $\log_e 1 = 0$.
4. $\int_2^4 \left(\frac{2}{5x-4} - \frac{3}{2-5x}\right) dx$. Hint: $\log_e 1 = 0$.

Answers

1. $2\log_e 5$ or $\log_e 25$ (using log law $a\log_e x = \log_e x^a$.
2. $\frac{2}{3}\log_e 3$.

3. $\frac{26}{3} - 2\ln 3$.
4. $\frac{2}{5}\ln\frac{8}{3} + \frac{3}{5}\ln\frac{9}{4}$.