ILS 1.1 INDICES

Index notation

Consider the following:

$3^4 = 3 \times 3 \times 3 \times 3 = 81$

$5^3 = 5 \times 5 \times 5 = 125$

$2^7 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 128$

In general

$$a^n = \underbrace{a \times a \times a \times a........a}_{n \text{ factors}}$$

The letter n in a^n is called one of three things:

- n is the **index** in a^n with a as the base
- n is the **exponent or power** to which the base a is raised
- n is the **logarithm**, with a as the base

When a number such as 125 is written in the form 5^3 we say that it is written as an exponential, or in index notation.

The manipulation of numbers that are written index notation is achieved by using the so called index laws.

Expressions in index form can be **multiplied** or **divided** only if they have the **same base.**

Index Laws

First Index Law

To **multiply** index expressions **add** the **indices.**

$2^3 \times 2^2 = 2 \cdot 2 \cdot 2 \times 2 \cdot 2 = 2^5$ (the product of 2 by itself 5 times)

Therefore $2^3 \times 2^2 = 2^{3+2} = 2^5$

In general:

$$a^m \times a^n = a^{m+n}$$ **First Index Law**

Second Index Law

To divide index expressions subtract the indices

$$3^5 \div 3^3 = \frac{3^5}{3^3} = \frac{3 \cdot 3 \cdot \cancel{3} \cdot \cancel{3} \cdot \cancel{3}}{\cancel{3} \cdot \cancel{3} \cdot \cancel{3}} = 3^2 \quad \text{(Cancelling three lots of 3)}$$

therefore $3^5 \div 3^3 = 3^{5-3} = 3^2$

In general $a^m \div a^n = a^{m-n}$ $\quad a \neq 0$ **Second Index Law**

Third Index Law

When an expression in index form is raised to a power, multiply the indices.

$\left(5^2\right)^3$ means $5^2 \times 5^2 \times 5^2$

From the First Index Law $\quad 5^2 \times 5^2 \times 5^2 = 5^{2+2+2} = 5^6$

therefore $\left(5^2\right)^3 = 5^{2\times3} = 5^6$

In general $\left(a^m\right)^n = a^{mn}$ **Third Index Law** also $\left(a^m b^p\right)^n = a^{mn} b^{pn}$

This is true for multiplication and division only, not addition or subtraction $(a+b)^n \neq a^n + b^n$

Examples

$x^5 \times x^6 = x^{11}$ using first index law

$y^8 \div y^3 = y^5$ using second index law

$\left(2x^2\right)^3 = 2^3 x^6 = 8x^6$ using third index law

Remember terms with different bases must be considered separately when using the index laws.

See Exercise 1

Zero Index

So far we have only considered expressions in which each index is a positive whole number.

The index laws apply just as well if the **index** is **zero** or **negative.**

Any expression with a **zero index** is equal to **1**.

Consider $2^3 \div 2^3$.

Using the second index law this is $2^{3-3} = 2^0$

But $2^3 \div 2^3$ is $\dfrac{2 \times 2 \times 2}{2 \times 2 \times 2} = \dfrac{\cancel{2} \times \cancel{2} \times \cancel{2}}{\cancel{2} \times \cancel{2} \times \cancel{2}} = 1$ (cancelling 3 factors of 2)

So $2^3 \div 2^3 = 2^0$ and $2^3 \div 2^3 = 1$

Therefore $2^0 = 1$

In general: $a^0 = 1 \quad a \neq 0$

Examples

$7^0 = 1 \qquad (xy)^0 = 1 \qquad \left(\tfrac{1}{2}\right)^0 = 1 \qquad \left(28x^2\right)^0 = 1$

Negative Index

Consider $2^0 \div 2^4$

$$2^0 \div 2^4 = 1 \div 2^4 = \frac{1}{2^4} \quad \text{because } 2^0 = 1$$

also $2^0 \div 2^4 = 2^{0-4} = 2^{-4}$ using the second index law

So $2^0 \div 2^4 = \frac{1}{2^4}$ and $2^0 \div 2^4 = 2^{-4}$ therefore $2^{-4} = \frac{1}{2^4}$

In general $a^{-n} = \frac{1}{a^n}$ and $\frac{1}{a^{-n}} = a^n \quad a \neq 0$

Examples

1. $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$ 2. $\frac{1}{x} = x^{-1}$ 3. $2y^{-1} = \frac{2}{y}$

5. $\frac{1}{3x^{-2}} = \frac{x^2}{3}$ 6. $\frac{1}{(-2a)^{-3}} = (-2a)^3$ 7. $5ab^{-4} = \frac{5a}{b^4}$

See Exercise 2

Summary of Index Laws

1. $a^m \times a^n = a^{m+n}$
2. $a^m \div a^n = a^{m-n} \quad a \neq 0$
3. $\left(a^m\right)^n = a^{mn}$
4. $a^0 = 1 \quad a \neq 0$
5. $a^{-n} = \frac{1}{a^n} \quad a \neq 0$

Combining index laws

Index laws are used to simplify complex expressions.

Example 1 $\left(4a^2b\right)^3 \div b^2 = 4^3a^6b^3 \div b^2 = 4^3a^6b$ using a combination of the second and third laws

Example 2 $\left(\frac{3a^3b}{c^2}\right)^2 \div \left(\frac{ab}{3c^{-2}}\right)^{-3} = \frac{3^2a^6b^2}{c^4} \div \frac{a^{-3}b^{-3}}{3^{-3}c^6}$ remove brackets

$= \frac{3^2a^6b^2}{c^4} \times \frac{3^{-3}c^6}{a^{-3}b^{-3}}$ divide, by inverting fraction and multiplying

$= 3^{2-3} \times a^{6+3} \times b^{2+3} \times c^{6-4}$ apply index laws

$= 3^{-1} \times a^9 \times b^5 \times c^2$

$= \frac{a^9b^5c^2}{3}$ write with positive indices

Example 3 Write $x^{-1}+x^{2}$ as a single fraction

$$x^{-1}+x^{2} = \frac{1}{x}+x^{2} = \frac{1+x^{3}}{x}$$ add using a common denominator, x.

Exercise 1

Simplify the following

(a) $c^5 \times c^3 \times c^7$ (b) $3\times 2^2 \times 2^3$ (c) $a^3 \times a^2b^3 \times ab^4$

(d) $3^6 \div 3^4$ (e) $a^8 \div a^3$ (f) $x^4y^6 \div x^2y^3$

(g) $\left(x^3\right)^4$ (h) $(x^m\, y^n)^5$

Exercise 2

Write with positive indices and evaluate if possible:

(a) x^{-6} (b) 250^0 (c) $3ab^{-5}$ (d) $(pq)^{-2}$

(e) $(5xy)^{-3}$ (f) $\frac{2y}{z^{-5}}$ (g) 2^{-5} (h) $(-2)^{-3}$

(i) $-\left(3^{-2}\right)$ (j) $2\times(-5)^{-2}$

Exercise 3

Simplify these expressions giving your answer in positive index form:

(a) $2a^3b^2 \times a^{-1}b^3$ (b) $\left(5x^{-2}y\right)^{-3}$ (c) $(3x^3\, y^{-1})^5$ (d) $(a^{-4}b^{-5})^{-2}$

(e) $\frac{a^2b^3c^{-4}}{a^4bc^5}$ (f) $\frac{a^7\times a^8\times a^3}{a^2\times a^5}$ (g) $x\left(x-x^{-1}\right)$ (h) $\frac{2^n\cdot 2^{3n}}{2^3}$

(i) $\frac{15a^2b}{3a^4b}\times\frac{4a^5b^2}{5a^3b^4}$ (j) 2^4-2^3

Answers

Exercise 1

(a) c^{15} (b) 3×2^5 (c) a^6b^7 (d) 3^2 (e) a^5 (f) x^2y^3 (g) x^{12} (h) $x^{5m}y^{5n}$

Exercise 2.

(a) $\frac{1}{x^6}$ (b) 1 (c) $\frac{3a}{b^5}$ (d) $\frac{1}{(pq)^2}$ (e) $\frac{1}{(5xy)^3}=\frac{1}{125x^3y^3}$ (f) $2yz^5$ (g) $\frac{1}{32}$ (h) $\frac{1}{-8}$ (i) $\frac{1}{-9}$ (j) $\frac{2}{25}$

Exercise 3.

(a) $2a^2b^5$ (b) $\frac{x^6}{5^3y^3}$ (c) $\frac{3^5x^{15}}{y^5}$ (d) a^8b^{10} (e) $\frac{b^2}{a^2c^9}$ (f) a^{11} (g) x^2-1

(h) 2^{4n-3} (i) $\frac{4}{b^2}$ (j) 2^3