Gradient of a Curve

The gradient at a point on a curve is the gradient of the tangent to the curve at that point.

For the curve $y = f(x)$ below, the gradient at point $P$ is the gradient of the line $AB$.

**Special cases:** horizontal and vertical lines

A line parallel to the x-axis with equation of the form $y = k$ ($k$ constant), has a gradient of zero.

As a line becomes closer to vertical it’s gradient gets larger and larger. A line parallel to the y-axis with equation of the form $x = c$ ($c$ constant) has a gradient which is undefined.
This gradient cannot be calculated - only one point on the line is known. The gradient of the line PQ can be calculated and this can be used to approximate the gradient of AB.

The gradient of \( PQ = \frac{f(x + h) - f(x)}{h} \).

As the value of \( h \) decreases (i.e., Q becomes closer to the point P), the approximation of the gradient is more accurate. The value of the gradient becomes most accurate as \( h \) approaches zero.

The gradient formula for the curve \( y = f(x) \) is defined as the derivative function

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}, \quad h \neq 0
\]

The derivative function \( f'(x) \) gives the slope of the tangent to the curve \( f(x) \) at any point \( x \).

**Example**

1. If the derivative function of \( f(x) = \frac{3}{x} \) is \( f'(x) = \frac{-3}{x^2} \), find the slope of the tangent to the curve at \( x = 4 \)

At \( x = 4 \), \( f'(x) = f''(4) \)

\[
f'(x) = \frac{-3}{4^2} = \frac{-3}{16}
\]
Exercises

1. If the derivative function for \( f(x) = x^3 - x \) is \( f'(x) = 3x^2 - 1 \), find the slope of the tangent to this curve at
   a) \( x = 2 \)
   b) \( x = 0 \)
   c) \( x = -9 \)

2. If the derivative function of \( f(x) = \sin(x) \) is \( f'(x) = \cos(x) \) find the gradient of \( y = \sin(x) \) at
   a) \( x = 0 \)
   b) \( x = \frac{\pi}{2} \)
   c) \( x = 3.5 \)

3. Determine \( \lim_{h \to 0} \frac{(x + h)^2 - x^2}{h} \) and hence find the slope of the tangent to the curve \( y = x^2 \) at
   a) \( x = 2 \)
   b) \( x = 0 \)
   c) \( x = -9 \)

Answers

1. a) 11
   b) -1
   c) 242

2. a) 1
   b) 0
   c) -0.94

3. \( \lim_{h \to 0} \frac{(x + h)^2 - x^2}{h} = 2x \)
   a) 4
   b) 0
   c) -18