FA1.2: FACTORISATION: QUADRATICS

Definition

The general form of a quadratic expression is:

\[ ax^2 + bx + c \quad a \neq 0 \]

where \( a, b, \) and \( c \) are constants and \( x \) is the unknown.

Expansion

Revision of expansion.

To expand an expression of the form \((a + b)(c + d)\), multiply each term in the first bracket by each term in the second bracket.

\[(x + 2)(x + 3) = x(x + 3) + 2(x + 3) = x^2 + 3x + 2x + 6\]

Simplify by combining like terms:

\[(x + 2)(x + 3) = x^2 + 5x + 6\]

Factorisation

Factorisation is the reverse of expansion.

\(x^2 + 5x + 6\) is expressed as the product of two factors, \((x + 2)\) and \((x + 3)\).

\[\begin{align*}
\text{factorisation} & \quad x^2 + 5x + 6 = (x + 2)(x + 3) \\
\text{expansion} & \quad \uparrow
\end{align*}\]

Note that:

- multiplying the first term in each bracket gives the term ‘\(x^2\)’ in the expression
- multiplying the last term in each bracket gives the constant term, (\(+6\)), in the expression
- the coefficient of the ‘\(x\)’ term is the sum of last term in each bracket (\(+2 + 3 = +5\)).

In general

**constant ‘\(c\)’ positive**

\[x^2 + bx + c = (x + m)(x + n)\]

and \[x^2 - bx + c = (x - m)(x - n)\]

**constant ‘\(c\)’ negative**

\[x^2 + bx - c = (x + m)(x - n) \quad m > n\]

\[x^2 - bx - c = (x + m)(x - n) \quad m < n \quad \text{where} \quad m\ \text{and} \ n \quad \geq \ 0.\]

Expressions with \(a = 1\)

If \(a = 1\) the expression becomes \(x^2 + bx + c\)
To factorise:

Find two numbers that:

- multiply to ‘c’ and
- add to ‘b’

Insert one number into each of the brackets \((x \ldots)(x \ldots)\).
The order is not important.

Note:

- If ‘c’ is positive, both numbers are positive or both numbers are negative.
  - If b is positive, both numbers are positive.
  - If b is negative, both numbers are negative.
- If c is negative, then one number is positive and the other negative;
  - If b is positive, then the larger number is positive.
  - If b is negative, then the larger number is negative.

Examples

Factorise the following.

1. \(x^2 + 9x + 14\)
   - the first term in each bracket must be ‘x’
   - Look for two numbers that
     - multiply to +14
     - add to +9
   - the numbers are +2 and +7
   \[x^2 + 9x + 14 = (x + 2)(x + 7)\]

2. \(y^2 - 7y + 12\)
   - the first term in each bracket must be ‘y’
   - Look for two numbers that
     - multiply to +12.
     - add to −7,
   - the numbers are −3 and −4
   \[y^2 - 7y + 12 = (y - 3)(y - 4)\]

3. \(p^2 - 5p - 14\)
   - the first term in each bracket must be ‘p’
   - Look for two numbers that
     - multiply to −14
     - add to −5,
     \(\text{(the larger number must be } - \text{ve) numbers are } - 7 \text{ and } +2\)
   \[p^2 - 5p - 14 = (p - 7)(p + 2)\]

4. \(a^2 + 6a - 7\)
   - the first term in each bracket must be ‘a’
   - Look for two numbers that
     - multiply to −7
     - add to +6
     \(\text{(the larger number must be } +\text{ve) numbers are } +7 \text{ and } -1\)
   \[a^2 + 6a - 7 = (a + 7)(a - 1)\]

Not all expressions of the form \(x^2 + bx + c\) can be factorised by this method. The factors may be of a more complicated form or there may be no real factors.
Discriminant

For a quadratic equation of the form $ax^2 + bx + c$ a method of checking for real factors is to calculate the value of $(b^2 - 4ac)$ (called the discriminant).

If $b^2 - 4ac < 0$ there are no real factors.

Example

Factorise (if possible) $a^2 + 6a + 12$

In this case:

'\[ b = +6 \quad c = +12 \]

\[ (b^2 - 4ac) = -12 \]

The discriminant is negative therefore $a^2 + 6a + 12$ has no real factors.

Exercise

Factorise the following (if possible).

1. $x^2 + 10x + 21$
2. $z^2 + 11z + 18$
3. $x^2 + 5x - 14$
4. $m^2 - m - 72$
5. $x^2 + 6x + 9$
6. $a^2 - 15a + 44$
7. $x^2 - 2x - 24$
8. $y^2 - 10y + 16$
9. $z^2 + 4z - 60$
10. $n^2 + 6n - 16$
11. $a^2 + 5a + 10$
12. $s^2 + 2s - 48$
13. $y^2 + 7y + 19$
14. $x^2 + 16x + 39$
15. $x^2 - 14x + 45$

Answers

1. $(x + 7)(x + 3)$
2. $(z + 9)(z + 2)$
3. $(x + 7)(x - 2)$
4. $(m - 9)(m + 8)$
5. $(x + 3)(x + 3)$
6. $(a - 11)(a - 4)$
7. $(x - 6)(x + 4)$
8. $(y - 2)(y - 8)$
9. $(z - 6)(z + 10)$
10. $(n - 2)(n + 8)$
11. no real factors
12. $(s + 8)(s - 6)$
13. no real factors
14. $(x + 13)(x + 3)$
15. $(x - 9)(x - 5)$
Expressions with  \( a \neq 1 \)

Expressions of the type \( ax^2 + bx + c \) can be factorised using a technique similar to that used for expressions of the type \( x^2 + bx + c \).

In this case the coefficient of \( x \), in at least one bracket, will not equal 1.

Consider the following product.

\[
(3x + 2)(2x + 1) = 6x^2 + 3x + 4x + 2
= 6x^2 + 7x + 2
\]

- multiplying the first term in each bracket gives the \('x^2'\) term (6\(x^2\)) in the expansion
- multiplying the last term in each bracket gives the constant term (2), in the expansion
- the coefficient of the \('x'\) term is the sum of the \('x'\) terms in the expansion. (+3\(x\) + 4\(x\) = +7\(x\)).

Examples

1. Factorise \( 2x^2 + 7x + 6 \)
   \[
   2x^2 + 7x + 6 = (2x\ldots)(x\ldots)
   2x^2 + 7x + 6 = (2x + 3)(x + 2)
   \]
   - \( 2x \times x = 2x^2 \)
   - find factors of +6
   - middle term positive so +6 and +1 or +3 and +2
   - try all combinations.
   - the one that gives +7\(x\) when the brackets are expanded is (2\(x + 3\)) and (\(x + 2\)).

2. Factorise \( 2x^2 - x - 6 \)
   \[
   2x^2 - x - 6 = (2x\ldots)(x\ldots)
   2x^2 - x - 6 = (2x + 3)(x - 2)
   \]
   - \( 2x \times x = 2x^2 \)
   - find factors of –6.
   - –6 & +1 or –6 & –1 or +3 & –2 or –3 & +2
   - try all combinations.
   - the one that gives –\(x\) when the brackets are expanded is (2\(x+3\)) and (\(x – 2\))

3. Factorise \( 3a^2 - 16a + 5 \)
   \[
   3a^2 - 16a + 5 = (3a\ldots)(a\ldots)
   3a^2 - 16a + 5 = (a - 5)(3a - 1)
   \]
   - \( 3a \times a = 3a^2 \)
   - find factors of +5
   - middle term negative so –5 and –1
   - try each combination
   - the one that gives –16\(a\) when the brackets are expanded is (\(a - 5\)) and (3\(a - 1\)).

4. Factorise \( 6y^2 + 16y - 6 \)
   \[
   6y^2 + 16y - 6 = 2(3y^2 + 8y - 3)
   = 2(3y\ldots)(y\ldots)
   6y^2 + 16y - 6 = 2(3y - 1)(y + 3)
   \]
   - Remove a common factor of 2.
   - This reduces the number of possible combinations.
   - \( 3y \times y = 3y^2 \)
   - find factors of –3
   - possibilities are +3 and –1 or –3 and +1
   - try all combinations
   - the one that gives +8\(y\) when the brackets are expanded is (3\(y - 1\)) and (\(y + 3\)).
To factorise a quadratic of the form $ax^2 + bx + c$

- search each term in the expression for a common factor (every term must have this factor).
- remove this from the expression then place before the brackets.
- factorise, if possible, the resulting quadratic expression

This approach is good provided the number of combinations of factors of ‘$a$’ and ‘$c$’ is limited.

For more complicated expressions of the type $ax^2 + bx + c$ there are other methods of factorisation which may be more suitable.

Exercise
Factorise the following

(1) $5x^2 + 13x + 6$  
(2) $2x^2 + x - 15$  
(3) $3m^2 - m - 2$
(4) $3y^2 - 10y + 8$  
(5) $2a^2 + 11a + 12$  
(6) $6x^2 - 11x + 5$

Answers

(1) $(5x + 3)(x + 2)$  
(2) $(2x - 5)(x + 3)$  
(3) $(3m + 2)(m - 1)$
(4) $(3y - 4)(y - 2)$  
(5) $(2a + 3)(a + 4)$  
(6) $(6x - 5)(x - 1)$