FA1.3: FACTORISATION: COMPLETING THE SQUARE

A quadratic expression may be factorised by the method of "completing the square".

A **perfect square** is of the form

\[(x + b)^2 = x^2 + 2bx + b^2\]

The first two terms of the right-hand side of the above equation are \((x^2 + 2bx)\). To make a perfect square or to **complete the square** on \(x^2 + 2bx\), the third term, \(b^2\) must be added.

Note that \(b^2 = \left(\frac{\text{half the coefficient of } x^2}{2}\right)^2\).

**Examples**

To complete the square on \(x^2 + 6x\) add \(\left(\frac{6}{2}\right)^2 = 9\)

\(x^2 + 6x\) are the first two terms of the complete square \(x^2 + 6x + 9\)

To complete the square on \(x^2 + 5x\) add \(\left(\frac{5}{2}\right)^2\)

\(x^2 + 5x\) are the first two terms of the complete square \(x^2 + 5x + \left(\frac{5}{2}\right)^2\)

To complete the square on \(x^2 - 10x\) add \(\left(\frac{10}{2}\right)^2 = 25\)

\(x^2 - 10x\) are the first two terms of the complete square \(x^2 - 10x + 25\)

Expressions of the form \(x^2 + bx + c\) can be factorised by completing the square on the first two terms and rewriting the expression in the form

\[\left(\frac{x + \frac{b}{2}}{2}\right)^2 - m^2\] where \(m^2 = -\left(\frac{b}{2}\right)^2 + c\)

The DOTS rule can then be used to factorise the expression.
Examples
Factorise by the method of completing the square

1. \( x^2 + 8x - 5 \):

\[
\begin{align*}
\quad x^2 + 8x - 5 &= (x^2 + 8x + 16) - 16 - 5 \\
&= (x + 4)^2 - 21 \\
&= (x + 4)^2 - (\sqrt{21})^2 \\
&= x^2 + 8x - 5 = (x + 4 + \sqrt{21})(x + 4 - \sqrt{21})
\end{align*}
\]

Halve the coefficient of \( x \) \( (8/2) = 4 \)
Square the result \( 4^2 = 16 \)
Add and subtract 16. (This does not change the expression)
Simplify
\( x^2 + 8x + 16 = (x + 4)^2 \)
\[ -16 - 5 = -21 \]
Write as the difference of two squares.
factorise using 'DOTS'

2. \( x^2 - 6x - 4 \)

\[
\begin{align*}
\quad x^2 - 6x - 4 &= (x^2 - 6x + 9) - 9 - 4 \\
&= (x - 3)^2 - 13 \\
&= (x - 3)^2 - (\sqrt{13})^2 \\
&= x^2 - 6x - 4 = (x - 3 + \sqrt{13})(x - 3 - \sqrt{13})
\end{align*}
\]

Halve the coefficient of \( x \) \( (6/2) = 3 \)
Square the result \( 3^2 = 9 \)
Add and subtract 9
Simplify
\( x^2 - 6x + 9 = (x - 3)^2 \), \( -16 - 4 = -21 \)
Write as the difference of two squares.
factorise using 'DOTS'

See Exercise 1

If the coefficient of \( x^2 \) is not 1, remove the coefficient as a common factor and then factorise by completing the square.

3. \( 2x^2 - 10x + 2 \)

\[
\begin{align*}
\quad 2x^2 - 10x + 2 &= 2(x^2 - 5x + 1) \\
&= 2 \left[ x^2 - 5x + \left( \frac{5}{2} \right)^2 - \left( \frac{5}{2} \right)^2 + 1 \right] \\
&= 2 \left[ \left( x - \frac{5}{2} \right)^2 - \frac{25}{4} + \frac{4}{4} \right] \\
&= 2 \left[ \left( x - \frac{5}{2} \right)^2 - \frac{21}{4} \right] \\
&= 2 \left[ \left( x - \frac{5}{2} \right)^2 - \left( \sqrt{\frac{21}{2}} \right)^2 \right] \\
&= 2 \left[ \left( x - \frac{5}{2} + \sqrt{\frac{21}{2}} \right) \left( x - \frac{5}{2} - \sqrt{\frac{21}{2}} \right) \right]
\end{align*}
\]

Remove the common factor of 2
Proceed as in example 1
Halve the coefficient of \( x \)
Square the result, then add and subtract the answer.
\( + \left( \frac{5}{2} \right)^2 - \left( \frac{5}{2} \right)^2 \)
simplify
write as difference of two squares
factorise using DOTS

See Exercise 2
Exercises

Exercise 1
Factorise by completing the square (if possible).

(a) \( x^2 + 6x - 5 \)   (b) \( x^2 + 4x + 2 \)   (c) \( a^2 - 8a - 2 \)
(e) \( x^2 - 5x + 5 \)   (f) \( y^2 + 3y + 4 \)   (g) \( a^2 - 5a - 1 \)

Exercise 2
(a) \( 4x^2 + 8x - 20 \)   (b) \( x^2 - 6x + 2 \)   (c) \( 12 - 4x - 2x^2 \)

Answers

Exercise 1
(a) \( (x + 3 + \sqrt{14})(x + 3 - \sqrt{14}) \)
(d) no real factors \( (x - 5/2 + \sqrt{5}/2)(x - 5/2 - \sqrt{5}/2) \)

Exercise 2
(a) \( 4(x + 1 + \sqrt{6})(x + 1 - \sqrt{6}) \)   (b) \( (x - 3 + \sqrt{7})(x - 3 - \sqrt{7}) \)   (g) \( -2(x + 1 + \sqrt{7})(x + 1 - \sqrt{7}) \)