DN1.5: CHAIN RULE

The ‘chain rule’ is used to differentiate a function which is the composition of two simpler functions.

If \( y = g(u) \) where \( u = h(x) \)

Then

\[
\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}
\]

Examples

1) Differentiate \( y = (2x - 1)^4 \)

Let \( u = 2x - 1 \), then \( y = u^4 \)

\[
\frac{du}{dx} = 2 \quad \text{and} \quad \frac{dy}{du} = 4u^3
\]

\[
\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}
\]

\[
= 4u^3 \times 2
\]

\[
= 8u^3
\]

\[
= 8(2x - 1)^3 \quad \text{[since } u = 2x - 1]\]

2) Find the derivative of \( y = \frac{1}{\sqrt[3]{5t^2 + 2t + 1}} \)

\( y = (5t^2 + 2t + 1)^{-\frac{1}{3}} \) \quad \text{[change to index form for easier differentiation]}

Let \( u = 5t^2 + 2t + 1 \), then \( y = \frac{1}{u} = u^{-\frac{1}{3}} \)

\[
\frac{du}{dt} = 10t + 2 \quad \text{and} \quad \frac{dy}{du} = -\frac{1}{3}u^{-\frac{4}{3}}
\]

\[
\frac{dy}{dt} = \frac{dy}{du} \times \frac{du}{dt}
\]

\[
= -\frac{1}{3}u^{-\frac{4}{3}} \times (10t + 2)
\]

\[
= -\frac{1}{3}(5t^2 + 2t + 1)^{-\frac{4}{3}} \times (10t + 2) \quad \text{[since } u = 5t^2 + 2t + 1]\]

\[
= -\frac{(10t + 2)}{3}(5t^2 + 2t + 1)^{-\frac{4}{3}} \quad \text{[after simplifying]}\]
3) Differentiate \( y = \sin 5x \)

\[
y = \sin 5x
\]

Let \( y = \sin(u) \) where \( u = 5x \)

\[
\frac{dy}{du} = \cos(u) \quad \text{and} \quad \frac{du}{dx} = 5
\]

Then

\[
\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}
\]

\[
= \cos(u), 5
\]

\[
= 5\cos(5x)
\]

4) If \( f(x) = \cos^3 x \) find \( f'(x) \)

\[
y = \cos^3 x = [\cos(x)]^3
\]

Let \( y = u^3 \) where \( u = \cos(x) \)

\[
\frac{dy}{du} = 3u^2 \quad \text{and} \quad \frac{du}{dx} = -\sin(x)
\]

Then \( f'(x) = \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} \)

\[
= 3u^2.(-\sin x)
\]

\[
= 3\cos^2 x.(-\sin x)
\]

\[
= -3\sin x \cos^2 x
\]

5) Differentiate \( (\log_4 x)^3 \)

Let \( y = u^3 \) where \( u = \log_4 v \) and \( v = 4x \) [The chain rule can be extended to three or more functions!!]

\[
\frac{dy}{du} = 3u^2 \quad \frac{du}{dv} = \frac{1}{v} \quad \text{and} \quad \frac{dv}{dx} = 4
\]

\[
\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dv} \times \frac{dv}{dx}
\]

\[
= 3u^2.\frac{1}{v} \times 4
\]

\[
= 3(\log_4 v)^2.\frac{1}{4x}
\]

\[
= 3(\log_4 4x)^2.\frac{1}{4x}
\]

\[
= \frac{3}{x}(\log_4 4x)^2
\]
Exercise

Find the derivatives of the following functions

1) \( y = \tan 3x \)
2) \( f(x) = \log_e \left( \frac{x}{2} \right) \)

3) \( y = \sin \left( \frac{\pi}{4} - 2x \right) \)
4) \( y = \cos^3 x \)

5) \( f(x) = e^{\sin x} \)
6) \( y = \sqrt{1 - \cos 5x} \)

Answers

1) \( 3\sec^2 3x \)
2) \( \frac{1}{x} \)

3) \( -2 \cos \left( \frac{\pi}{4} - 2x \right) \)
4) \( -2 \sin x \cos x \)

5) \( e^{\sin x} \cos x \)
6) \( \frac{5 \sin 5x}{2\sqrt{1 - \cos 5x}} \)