IN1.6: APPLICATIONS OF INTEGRATION
AREA UNDER A CURVE

Calculation of Areas
The definite integral \( \int_a^b f(x) \, dx \) gives a measure of the signed area between the graph of \( f(x) \) and the \( x \)-axis between the values of \( x = a \) and \( x = b \).

If the area is above the \( x \)-axis, the definite integral is positive.

If the area is below the \( x \)-axis, the definite integral is negative.

The actual area is the absolute value of the signed area.

\[
A = \left| \int_a^b f(x) \, dx \right|
\]

Note: If the curve crosses the \( x \)-axis, the areas above and below the \( x \)-axis must be calculated separately.

Examples

1.
Given the function \( f(x) = x^2 - 1 \), find the area between the curve and the \( x \)-axis, for the values \( x = 0 \) and \( x = 3 \).

Sketch the graph of \( y = x^2 - 1 \) to determine the area to be calculated

| \( y \)-intercept: \( -1 \) |
| \( x \)-intercepts: \( \pm 1 \) |

Limits of integration: \( x = 0 \) to \( x = 3 \).

Area from \( x = 0 \) to \( x = 1 \) is below the \( x \)-axis. Area from \( x = 1 \) to \( x = 3 \) is above the \( x \)-axis. Integrate each area separately.

\[
A = \left| \int_0^1 (x^2 - 1) \, dx \right| + \left| \int_1^3 (x^2 - 1) \, dx \right|
\]

\[
= \left| \left[ \frac{x^3}{3} - x \right]_0^1 \right| + \left| \left[ \frac{x^3}{3} - x \right]_1^3 \right|
\]

\[
= \left| \left[ \frac{1}{3} - 1 \right] - 0 \right| + \left| \left[ \frac{27}{3} - 3 \right] - \left[ \frac{1}{3} - 1 \right] \right|
\]

\[
= \frac{2}{3} + \left( 6 + \frac{2}{3} \right)
\]

Area = \( 7 \frac{1}{3} \) units
Given the function \( f(x) = x^2 + 2 \), find the area between the curve, the \( x \)-axis, and the lines \( x = -1 \) and \( x = 1 \).

Sketch the graph of and \( y = x^2 + 2 \) to determine the area to be calculated.

- \( y \)-intercept: 2
- \( x \)-intercepts: no \( x \)-intercepts

Limits of integration:
\[ x = -1 \text{ to } x = 1 \]

\[
A = \left| \int_{-1}^{1} (x^2 + 2) \, dx \right| = \left| \left[ \frac{x^3}{3} + 2x \right]_{-1}^{1} \right|
\]
\[
= \left[ \frac{1}{3} + 2 \right] - \left[ -\frac{1}{3} - 2 \right] = 4 \frac{2}{3} \text{ units}
\]

Exercises

Exercise 1
(a) Find the area enclosed by the curve \( y = 6x - 2x^2 \) and the \( x \)-axis.
(b) Calculate the area bounded by the curve \( y = x^2 - 9 \) and the \( x \)-axis.
(c) Find the area bounded by the curve \( y = \frac{2}{x} \), the \( x \)-axis and the lines \( x = 2 \) and \( x = 4 \)
(d) Find the area bounded by the curve \( y = \sin x \) the \( x \)-axis and the line \( x = \frac{\pi}{2} \)

Exercise 2
(a) Find the total area of the regions bounded by the curve \( y = \cos x \) and the \( x \)-axis between \( x = 0 \) and \( x = \pi \).
(b) Find the area bounded by the curve \( y = x^2 - 3x + 2 \) and the \( x \)-axis between \( x = 0 \) and \( x = 2 \).

Answers

1(a) 9 units, (b) 36 units, (c) \( 2 \ln 2 \), (d) 1 unit
2(a) 2 units, (b) 1 unit