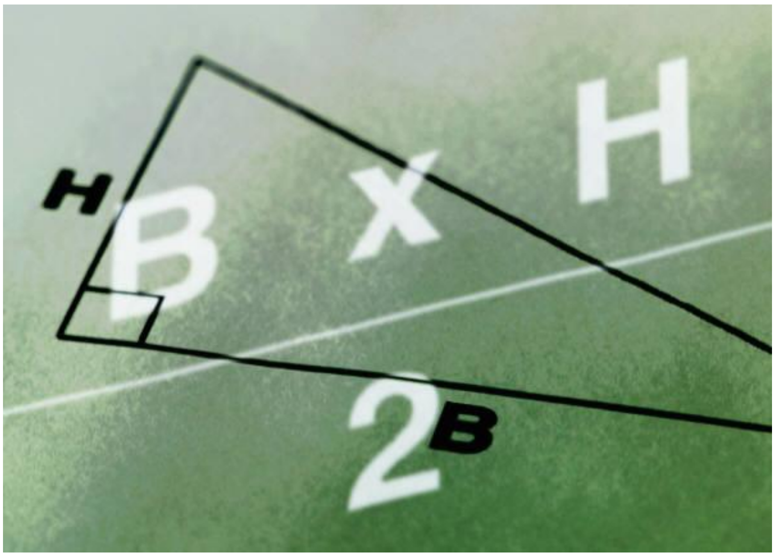
SUMMER

KNOWHOW
STUDY AND LEARNING CENTRE

RMIT UNIVERSITY

Algebra Skills



Contents

ALGEBRAIC OPERATIONS

Like Terms

Like terms contain exactly the same pronumerals (letters, variables)

Like Terms	*Unlike terms*
3x, 5x	3x, 4y
2a, -3a	3a, 3
$3m^2$, m^2	$3m^2$, 3m
2ab, 3ab	$2a^2b$, $3ab^2$
3ef, 5fe	3ef, 7fg

NB: Order is unimportant but alphabetical order of pronumerals is conventional

See Exercise 1

Addition and Subtraction

Only like terms may be added or subtracted.

1. $7e + 10e = (7 + 10)e$ [eventually it is possible to just THINK the second step and go straight to the answer]
   $= 17e$

2. $3x^2 - x^2 - 4x^2 = (3 - 1 - 4)x^2$
   $= -2x^2$

3. $3m - 4n + 6m + n = (3 + 6)m + (-4 + 1)n$
   $= 9m - 3n$

4. $3a - b - 5a + 4ab - 3b + ab = (3 - 5)a + (-1-3)b + (4 + 1)ab$
   $= -2a - 4b + 5ab$

5. $3x - x^2$ cannot be simplified

6. $p + 2p - 3 = 3p - 3$

7. $8uv + 3u - 10vu = -2uv + 3u$

8. $6rx^2 - 2rs^2$ cannot be simplified

See Exercise 2

Multiplication

Consider the sign (positive or negative) of the answers to the following simple multiplication problems

$2 \times 3 = 6$	positive x positive →→	positive
$-2 \times 3 = -6$	negative x positive →→	negative
$2 \times -3 = -6$	positive x negative →→	negative
$-2 \times -3 = 6$	negative x negative →→	positive

Terms can be multiplied whether they are like or unlike terms.

When multiplying two or more terms consider

- the sign of the answer
- the product of the numbers

and use an index to show how many factors of each pronumeral

1. $(-4) \times (-3b) = 12b$
2. $-2 \times 6y = -12y$
3. $2e \times (-5e^2) = -10e^3$ [three factors of 'e' altogether]
4. $(-2u^2v) \times (-4v) = 8u^2 v^2$
5. $-3pq \times (-2q) \times p = 6p^2 q^2$

In algebra a fraction line means to divide eg. ½ = 1 ÷ 2, ¾ = 3 ÷ 4

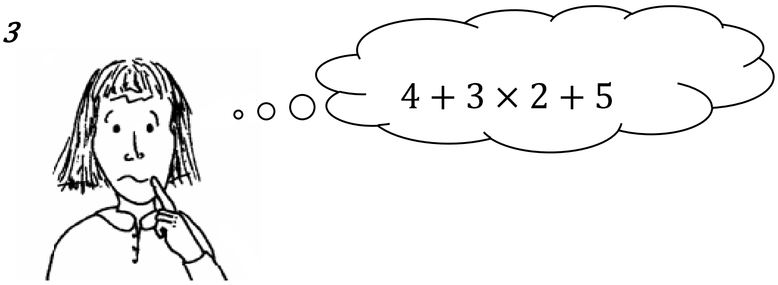
When dividing algebraic expressions

- rewrite as a fraction if necessary
- expand any powers
- establish the sign of the answer
- cancel factors
- write the answer in simplified form

1. $-12wyz \div 3yz = \dfrac{-12wyz}{3yz} = -4w$
2. $-2m^2n \div 6mn^2 = \dfrac{-2m^2n}{6mn^2} = \dfrac{-2mmn}{6mnn} = -\dfrac{m}{3n}$
3. $6a^2 \times 4b \div 12ab = \dfrac{6aa \times 4b}{12ab} = 2a$

NB: $\dfrac{m+2}{4m}$ CANNOT be simplified!!!

See Exercise 3



Order of Operations

Consider **4 + 3 × 2 + 5.** What is the answer?

$4 + 3 \times 2 + 5 \rightarrow 7 \times 7 = 49?$

$4 + 3 \times 2 + 5 \rightarrow 4 + 3 \times 7 = 25?$

$4 + 3 \times 2 + 5 \rightarrow 7 \times 2 + 5 = 19?$

To avoid such confusion we perform operations in the following order

1.	**B**rackets	**B**
2.	**I**ndices	**I**
3.	**M**ultiplication and **D**ivision from left to right	**M**
		D
4.	**A**ddition and **S**ubtraction from left to right	**A**
		S

To make it easier to remember the rule for order of operations is abbreviated to **B I M D A S**

Using this rule $4 + 3 \times 2 + 5 = 4 + 6 + 5 = 15$!!

Examples:

1. $3 \times 2 + 4 = 6 + 4 = 10$
2. $3 + 2 \times 4 = 3 + 8 = 11$
3. $(3 + 2) \times 4 = 5 \times 4 = 20$
4. $3 - 2^2 = 3 - 4 = -1$
5. $(3 - 2)^2 = 1^2 = 1$
6. $3^2 - 2^2 = 3 \times 3 - 2 \times 2 = 9 - 4 = 5$
7. $3st - 3s \times 4t = 3st - 12st = -9st$

See Exercise 4

Exercise 1

Which of the five terms on the right is a like term with the term on the left?

1)	$3x$	3	$2x$	$3x^2$	$2xy$	$4x^2a$
2)	$2ab$	$2a$	$2b$	$3x^2$	$6abc$	$12ab$
3)	$2x^2$	x	$2x$	$5x^2$	$4x$	$4x^3$
4)	$3xy^2$	3	$3xy$	$3x^2$	xy^2	$3y^2$
5)	$2m^2n$	n	mn^2	$8mn$	$2m^2$	$4nm^2$
6)	$4ab^2c$	4	$2ab^2c$	$4abc$	$8b^2c$	$4cba$

Exercise 2

Simplify each of the following.

1. a) $5x + 3x$ b) $12x - 7x$
   c) $15xy + 5xy$ d) $11mn - 5mn$
   e) $10abc - 3bca$ f) $10m - 22m$
   g) $6x + 3x + 4x$ h) $4ab + 5ab - 2ab$
   i) $m + 2m - 9m$ j) $14xy - 4xy + 2xy$

2. a) $13x + 4 - 3x - 1$ b) $10mn + 5m + 12mn + 6m$
   c) $3xy^2 + 2xy + 5xy^2 + 3xy$ d) $5xy + 6m - 2xy - 2m$
   e) $x + y + 2x - y$ f) $3a + 5b - a - 6b$
   g) $x + 4y - x - 2y$ h) $7x - 4m - 5x - 3m$
   i) $4x - 5x - 3y + 5x$ j) $9mn - 3m - n + 4m^2$

Exercise 3

Simplify the following algebraic expressions

1. a) $5 \times 2k$ b) $4a \times 3ab$
   c) $y \times 3y$ d) $4m \times (-3mn)$
   e) $m \times 3p \times 5$ f) $2ab \times 3bc \times (-4)$
   g) $2ab^2 \times 3ac$ h) $4m \times (-5kmp)$

2. a) $18ef \div 6f$ b) $-100uvw \div 100w$
   c) $24gh^2 \div 8gh$ d) $3m^2n \div 12mn^2$
   e) $rs \times 2st \div 2s$ f) $3jk \times 12km \div 9jkm$
   g) $10p \times 3pq \div 16pq$ h) $4yz \times 5w^2z \div 10wy$

Exercise 4

Simplify

1) $18 + (3 \times -5)$
2) $3 \times -4 + (8 \times 2)$
3) $10 - 5^2 + 3$
4) $(10 - 5)^2 + 3$
5) $10^2 - 5^2$
6) $(10 - 5)^2$
7) $3m + 2 \times 3m$
8) $6ab - 3a \times 4b$
9) $16gh - 4gh \times 4$
10) $9b - 3b \times 2k + 2k \times b$

Answers

Exercise 1

1. $2x$ 2. $12ab$ 3. $5x^2$ 4. xy^2 5. $4nm^2$ 6. $2ab^2c$

Exercise 2

1. a. $8x$ b. $5x$ c. $20xy$ d. $6mn$ e. $7abc$
   f. $-12m$ g. $13x$ h. $7ab$ i. $-6m$ j. $12xy$
2. a. $10x + 3$ b. $22mn + 11m$
   c. $8xy^2 + 5xy$ d. $3xy + 4m$
   e. $3x$ f. $2a - b$
   g. $2y$ h. $2x - 7m$
   i. $4x - 3y$ j. $9mn - 3m - n + 4m^2$

Exercise 3

1. a. $10k$ b. $12a^2b$ c. $3y^2$ d. $-12m^2n$ e. $15mp$ f. $-24ab^2c$ g. $6a^2b^2c$ h. $-20km^2p$

2. a. $3e$ b. $-uv$ c. $3h$ d. $\frac{m}{4n}$ e. rst f. $4k$ g. $\frac{15p}{8}$ h. $2wz^2$

Exercise 4

1. 3 2. 4 3. -12 4. 28 5. 75
6. 25 7. $9m$ 8. $-6ab$ 9. 0 10. $9b - 4bk$

ALGEBRAIC SUBSTITUTION

The process of replacing a *pronumeral* (or variable or letter) in an expression or formula with a number is called *substitution.*

After substitution it is possible to *evaluate* the algebraic expression.

Examples

1. *Evaluate 3p if p = -5*

$$3p = 3 \times p$$
$$= 3 \times -5$$
$$= -15$$

2. *Evaluate* $\frac{a+5}{b}$ *if a = -9 and b = 2*

$$\frac{a+5}{b} = \frac{-9+5}{2}$$
$$= \frac{-4}{2}$$
$$= -2$$

3. *Evaluate* w^2 *- 2z if w = -1, z = 5*

$$w^2 - 2z = (-1)^2 - 2 \times 5$$
$$= 1 - 10$$
$$= -9$$

4. *Use the formula* $C = \frac{5(F-32)}{9}$ *to convert a temperature of 100° in the Fahrenheit system to Centigrade.*

$$C = \frac{5(F-32)}{9}$$
$$= \frac{5(100-32)}{9}$$
$$= \frac{5(68)}{9}$$
$$= 37.78$$

Exercises

1. Evaluate the following

a) $-4k$ if $k = 7$

b) $2mn$ if $m = 4, n = -2$

c) $e^2 - 5$ if $e = 2$

d) $5 - b + b^2$ if $b = 3$

e) $2k^2 + 4$ if $k = -6$

f) $-3ab^2$ if $a = 4, b = 2$

g) $\frac{n}{4} + 2$ if $n = 10$

h) $\frac{u}{5v}$ if $u = -20, v = 2$

2. Evaluate the following if $a = -1, b = 6, c = 3, m = -2, n = 2$

a) $3a^2 - 7$

b) $3a - b^2$

c) $(2m + 1)^2$

d) $3(a^2 + 4)$

e) $\frac{2m}{n}$

f) $(m - n)^2$

Answers

1. a) -28 b) -16 c) -1 d) 11 e) 76 f) -48 g) 4.5 h) -2

2. a) -4 b) -39 c) 9 d) 15 e) -2 f) 16

REMOVING BRACKETS

To expand brackets like $a(b + c)$ ie $a \times (b + c)$ we use the rule:

$$a \times (b + c) = a \times b + a \times c$$

$$\text{or } a(b+c) = ab + ac$$

so $2 \times (3 + 5) = 2 \times 3 + 2 \times 5$

Examples

1. $3(x + 2) = 3 \times x + 3 \times 2$
   $= 3x + 6$

2. $-5(m + 4) = (-5) \times m + (-5) \times 4$
   $= -5m + (-20)$
   $= -5m - 20$

3. $-2(p - 7) = -2p + 14$

4. $e(e + 2) = e^2 + 2e$ [leaving out the second step]

5. $-(4p - 3) = (-1)(4p - 3)$ [nb: $-x = (-1)x$]
   $= -4p + 3$

6. $4(x - 3) + 2(x + 1) = 4x - 12 + 2x + 2$
   $= 6x - 10$ [after simplifying]

7. $6k(2k + 3) - 3k(k - 3) = 12k^2 + 18k - 3k^2 + 9k$
   $= 9k^2 + 27k$

See Exercise 1

Binomial Products

A *binomial product* looks like this: $(m + 2)(m + 3)$.

It is the product of two *binomial* (two term) expressions. To expand the two brackets, multiply each term in one bracket by each term in the other bracket and simplify if possible.

$$\begin{aligned}(m + 2)(m + 3) &= m(m + 3) + 2(m + 3) \\ &= m \times m + 3 \times m + 2 \times m + 2 \times 3 \\ &= m^2 + 3m + 2m + 6 \\ &= m^2 + 5m + 6\end{aligned}$$

These steps can be shown algebraically as:

$$(A + B)(C + D) = AC + AD + BC + BD$$

Important special cases of the expansion are:

$$\begin{aligned}(a + b)^2 &= (a + b)(a + b) \\ &= a^2 + ab + ab + b^2 \\ &= a^2 + 2ab + b^2\end{aligned}$$

$$\begin{aligned}(a - b)^2 &= (a - b)(a - b) \\ &= a^2 - ab - ab + b^2 \\ &= a^2 - 2ab + b^2\end{aligned}$$

Examples

1\. $(2f + 3)(g + 5h) = 2f \times g + 3 \times g + 2f \times 5h + 3 \times 5h$
$= 2fg + 3g + 10fh + 15h$ [sometimes it is not possible to simplify further]

2 $(p - 2)(p - 7) = p^2 - 2p - 7p + 14$
$= p^2 - 9p + 14$

3\. $(2w + 3)(w - 4) = 2w^2 + 3w - 8w - 12$
$= 2w^2 - 5w - 12$

4\. $(4r - 3s)^2 = (4r - 3s)(4r - 3s)$
$= 16r^2 - 12rs - 12rs + 9s^2$
$= 16r^2 - 24rs + 9s^2$

See Exercise 2

Exercise 1

1 Remove the brackets

a) $2(x - 5)$ b) $3(6 - b)$ c) $(b + 5)c$
d) $2x(2a + 3b)$ e) $-6(p - 2q + 4)$ f) $3y(yz - 2y + 1)$

2 Expand and simplify

a) $3(x + 1) + 2(x + 2)$ b) $2(p - 1) - (p - 3)$
c) $2(4m - 3) - 5(9 - 2m)$ d) $2w(3w + 1) + 3w(2w - 5)$

3 Expand the following expressions and simplify where possible remembering the correct order of operations

a) $4(2m - 3) + 8$ b) $9 - 3(4b + 3)$
c) $1 - 4(x - 1)$ d) $8m - 3(1 - 2m) + 6$

Exercise 2

1 Expand the following binomial products

a) $(x + 5)(x + 3)$ b) $(u - 5)(u + 3)$
c) $(2c - d)(3c + 2d)$ d) $(3a - 2b)(3a + 2b)$

2 Remove the brackets

a) $(d + 3)^2$ b) $(4f - 3gh)^2$

Answers

Exercise 1

1 a) $2x - 10$ b) $18 - 3b$ c) $bc + 5c$ d) $4ax + 6bx$
e) $-6p + 12q - 24$ f) $3y^2z - 6y^2 + 3y$
2 a) $5x + 7$ b) $p + 1$ c) $18m - 51$ d) $12w^2 - 13w$
3 a) $8m - 4$ b) $-12b$ c) $5 - 4x$ d) $14m + 3$

Exercise 2

1 a) $x^2 + 8x + 15$ b) $u^2 - 2u - 15$ c) $6c^2 + cd - 2d^2$ d) $9a^2 - 4b^2$
2 a) $d^2 + 6d + 9$ b) $16f^2 - 24fgh + 9g^2h^2$

FACTORISATION: COMMON FACTORS

Expansion of brackets (or *removing brackets*) in an algebraic expression is done by multiplying *all* the terms inside the brackets by the term(s) outside the brackets.
e.g

$5a(3y - 2x) = 15ay - 10ax$
Each term inside the brackets has been multiplied by $5a$.

Factorisation is the reverse of expansion. To *factorise* a number or algebraic expression means to write the number or expression as a *product* (multiplication).

Examples

1. $-2xyz$ has factors -2, x, y, and z
2. $7(a + b)$ has factors 7 and $(a + b)$
3. $3(x - y)$ has factors 3 and $(x - y)$
4. $x(3a + 2b)$ has factors x and $(3a + 2b)$
5. $2p(2r+1)(s+4)$ has factors 2, p, $(2r+1)$ and $(s+4)$

Expansion means *removing brackets*
Factorisation means *inserting brackets*

Factorisation by removing a common factor

- Search each term in the expression for a common factor (**every term** must have this factor)
- There may be several common factors. Search until you have found all of them
- If there is more than one common factor multiply them to give highest common factor. (HCF)
- The HCF is placed before the bracket
- The terms inside the bracket are found by dividing each term by the HCF.

Examples

1. *Factorise* $5y + 10$

5 *times y equals* $5y$

$5y + 10 = 5 \times y + 5 \times 2$ [factorise, common factor 5]

common factor of 5

5 *times* 2 *equals* 10

$5y + 10 = 5(y + 2)$ [5 before brackets, (y+2) inside brackets]

2. *Factorise* $3x + 3y$

$3x + 3y = 3 \times x + 3 \times y$ [common factor 3]

$= 3(x + y)$

Similarly

3. $p^2 + p = p \times p + p \times 1$ [common factor p]
$= p(p + 1)$

4. $7y^2 + 7y = 7y \times y + 7y \times 1$ [common factors 7 and y HCF=7y]
$= 7y(y + 1)$

5. $2abc - 12ac = 2 \times a \times b \times c - 2 \times 6 \times a \times c$
$= 2ac(b - 6)$ [common factor 2, a, and c. HCF = 2ac]

See Exercise 1

A common factor may be negative.

Examples

1. $-2a - 2b = (-2) \times a + (-2) \times b$ [common factor – 2]
$= -2(a + b)$

2. $-3x + 6xy = (-3x) \times 1 - (-3x) \times 2y$ [HCF = – 3x]
$= -3x(1 - 2y)$

See Exercise 2

Exercise 1

Factorise the following (if possible)

a) $3x + 3y$ b) $8a - 8b + 8c$ c) $xy - 5x$ d) $7x + 21y$
e) $5m - 2n$ f) $5mn - 10n$ g) $3m^2 - 3mnp$ h) $12m^2 - 18mn$
i) $2ab^2c + 6abc^2$ j) $5xyz - x^2yz^2 + 10x$

Exercise 2

Factorise the following by removing a negative factor.

a) $-3x - 6y$ b) $-15xy + 25xz$ c) $-6xyz - 15yz - 3xy^2z$

Answers

Exercise 1

a) $3(x + y)$ b) $8(a - b + c)$ c) $x(y - 5)$ d) $7(x + 3y)$
e) no factors f) $5n(m - 2)$ g) $3m(m - np)$ h) $6m(2m - 3n)$
i) $2abc(b + 3c)$ j) $x(5yz - xyz^2 + 10)$

Exercise 2

a) $-3(x + 2y)$ b) $-5x(3y - 5z)$ c) $-3yz(2x + 5 + xy)$

FACTORISATION: QUADRATICS(i)

Definition

The general form of a quadratic expression is

$$ax^2 + bx + c, \quad a \neq 0$$

where a, b, c are constants and x is the unknown

Review of Expansion

To expand an expression of the form $(a + b)(c + d)$, multiply each term in the first bracket by each term in the second bracket.

For example : $(x + 2)(x + 3) = x(x + 3) + 2(x + 3)$

$= x^2 + 3x + 2x + 6$ [Simplify by combining like terms]

$= x^2 + 5x + 6$

Factorisation of expressions with a = 1

If $a = 1$ the expression becomes $x^2 + bx + c$

Factorisation is the reverse of expansion.

$x^2 + 5x + 6$ is expressed as the product of two factors, $(x + 2)$ and $(x + 3)$

factorisation →

$$x^2 + 5x + 6 = (x + 2)(x + 3)$$

expansion ←

Note that:

- multiplying the first term in each bracket gives the term 'x^2' in the expression
- multiplying the last term in each bracket gives the constant term, +6, in the expression
- the coefficient of the 'x' term is the sum of last term in each bracket $(+2 + 3 = +5)$.

In general

- if the constant 'c' is positive
  $x^2 + bx + c = (x + m)(x + n)$ and $x^2 - bx + c = (x - m)(x - n)$
- if the constant 'c' is negative
  $x^2 + bx - c = (x + m)(x - n)$, $m > n$ and $x^2 - bx - c = (x - m)(x + n)$ $m < n$ and $m, n \geq 0$.

To factorise:

Find two numbers *m* and *n* that:

- multiply to '*c*' and
- add to '*b*'

Insert into brackets (x ....)(x ....)
Order is not important.

If 'c' is positive, both numbers are positive or both numbers are negative.
If b is positive, both numbers are positive.
If b is negative, both numbers are negative.

If c is negative, then one number is positive and the other negative;
If b is positive, then the larger number is positive.
If b is negative, then the larger number is negative.

Examples

Factorise the following.

1. $x^2 + 9x + 14 = (x\dots\ )(x\ \dots\ )$
$= (x + 2)(x + 7)$

[the first term in each bracket must be 'x']
[Find m and n so that $m \times n = 14$ and $m + n = 9$
$\Rightarrow$ $m = +2$ and $n = +7$]

2. $y^2 - 7y + 12 = (y\ \dots)(y\ \dots\ )$
$= (y - 3)(y - 4)$

[the first term in each bracket must be 'y']
[Find m and n so that $m \times n = 12$ and $m + n = -7$
$\Rightarrow$ $m = -3$ and $n = -4$]

3. $p^2 - 5p - 24 = (p\dots\ )(p\dots\ )$
$= (p - 8)(p + 3)$

[the first term in each bracket must be 'p']
[Find m and n so that $m \times n = -24$ and $m + n = -5$
$\Rightarrow$ $m = -8$ and $n = +3$ since larger number must be -ve]

4. $a^2 + 6a - 7 = (a\dots\ )(\ a\dots\ )$
$= (a + 7)(a - 1)$

[the first term in each bracket must be 'a']
[Find m and n so that $m \times n = -7$ and $m + n = +6$
$\Rightarrow$ $m = 7$ and $n = -1$ since larger number must be +ve]

Some expressions may have irrational factors or no factors at all.
The discriminant can be used to check the nature of the factors of a quadratic expression

For a quadratic expression $ax^2 + bx + c$, **$b^2 - 4ac$** is called the **discriminant**

- $b^2 - 4ac > 0$ and a perfect square $\Rightarrow$ rational factors (can be factorised as above)
- $b^2 - 4ac > 0$ but not a perfect square $\Rightarrow$ irrational factors (use quadratic formula)
- $b^2 - 4ac = 0$ $\Rightarrow$ $x^2 + bx + c = (x + m)^2$
- $b^2 - 4ac < 0$ $\Rightarrow$ there are no real factors

Example

Factorise (if possible) $a^2 + 6a + 12$

$a = 1, b = 6, c = 12$
and $b^2 - 4ac = 6^2 - 4 \times 1 \times 12 = -12$
The discriminant is negative, therefore $a^2 + 6a + 12$ has no real factors.

Exercise
Factorise the following (if possible).

1. $x^2 + 10x + 21$
2. $m^2 - m - 72$
3. $x^2 - 2x - 24$
4. $n^2 + 6n + 16$
5. $y^2 + 20y + 51$
6. $a^2 - 15a + 44$

Answers
1. $(x + 7)(x + 3)$ 2. $(m - 9)(m + 8)$ 3. $(x - 6)(x + 4)$ 4. no real factors 5. $(y + 17)(y + 3)$ 6. $(a - 11)(a - 4)$

FACTORISATION: PERFECT SQUARES

Perfect squares

Examples of perfect squares are 5^2 , x^2 , a^2 , b^2, $(xy)^2$ and $(a + b)^2$

$(a + b)^2$ can be expanded as follows

$(a + b)^2 = a^2 + 2ab + b^2$

Similarly $(a - b)^2 = a^2 - 2ab + b^2$

$(a + b)^2$ **does not equal** $a^2 + b^2$	and	$(a - b)^2$ **does not equal** $a^2 - b^2$

Note that the expansion gives three terms:

- the first and last terms of the expansion must be positive
- the middle term is twice the product of the first and last terms
- the middle term may be positive or negative

The rule for expanding perfect squares can be used in reverse for factorising perfect squares.

Examples

Check each of the following expressions and, if it is a perfect square, state the perfect square.

1. $x^2 + 14x + 49$
   $= (x + 7)^2$
   - first term is the square of x
   - last term is positive and is the square of 7
   - middle term $= 2 \times x \times (\pm 7)$

2. $y^2 - 20y + 25$
   is not a perfect square
   - first term is the square of y
   - last term is positive and is the square of ± 5
   - BUT middle term does not equal $2 \times y \times (\pm 5)$

3. $4a^2 - 12a - 9$
   is not a perfect square
   - first term is the square of 2a
   - last term is negative (9 is a perfect square but -9 is not)

4. $100x^2 - 180x + 81$
   $= (10x - 9)^2$
   - first term is $(10x)^2$
   - last term is $(\pm 9)^2$
   - middle term is $2 \times 10x \times (\pm 9)$

5. $50x^2 + 80x + 32$
   $= 2(25x^2 + 40x + 16)$
   $= 2(5x + 4)^2$
   - Not a perfect square but can be factorised
   - And $25x^2 + 40x + 16$ is a perfect square

Exercise

Check each of the following expressions. If it is a perfect square factorise.

1. $a^2 + 2a + 1$
2. $x^2 - 4x + 4$
3. $25x^2 - 10x + 1$
4. $81x^2 + 108x + 36$
5. $9a^2 - 24a - 16$
6. $2x^2 + 8x + 8$

Answers

1. $(a + 1)^2$
2. $(x - 2)^2$
3. $(5x - 1)^2$
4. $(9x + 6)$
5. Not a perfect square
6. $2(x + 2)^2$

FACTORISATION: DIFFERENCE OF TWO SQUARES

Consider the following expansion:

$(x + 5)(x - 5) = x^2 + 5x - 5x + 25$

$= x^2 - 25$

Note that:

- the terms in the brackets differ only in the sign of the second term
- the expansion is the difference of two terms, both of which are perfect squares

In general:

$$(a + b)(a - b) = a^2 - b^2$$

This can be used to factorise expressions of the form $a^2 - b^2$

Examples

1. $a^2 - 36 = (a)^2 - (6)^2$ [expression is the difference of two squares]
   $= (a + 6)(a - 6)$ ['DOTS' rule (order of the terms on right hand side is not important)]

2. $4x^2 - y^2 = (2x)^2 - (y)^2$
   $= (2x + y)(2x - y)$ ['DOTS' rule]

3. $3x^2 - 48 = 3(x^2 - 16)$ [not yet a difference of two squares so first take out a common factor of 3]
   $= 3[(x)^2 - (4)^2]$
   $= 3(x + 4)(x - 4)$

4. $(x + 2)^2 - 9 = (x + 2)^2 - (3)^2$
   $= (x + 2 + 3)(x + 2 - 3)$ ['DOTS' rule]
   $= (x + 5)(x - 1)$ [Simplify]

5. $y^2 + 36$ [this is not the *difference* of two squares and has no real factors]

Exercise

Factorise the following using the 'DOTS' rule (if possible).

1. $x^2 - 4$
2. $49 + x^2$
3. $121x^2 - 49y^2$
4. $a^2b^2 - 25$
5. $5x^2 - 20$
6. $x^2y^3 - 36y$
7. $(x + 5)^2 - 36$
8. $(a + 1)^2 - (b - 2)^2$

Answers

1. $(x + 2)(x - 2)$
2. Does not factorise
3. $(11x + 7y)(11x - 7y)$
4. $(ab + 5)(ab - 5)$
5. $5(x + 2)(x - 2)$
6. $y(xy + 6)(xy - 6)$
7. $(x + 11)(x - 1)$
8. $(a + b - 1)(a - b + 3)$

FACTORISATION: QUADRATICS(ii)

Consider the expansion of the product $(3x + 2)(2x + 1)$:

$$(3x + 2)(2x + 1) = 6x^2 + 3x + 4x + 2$$
$$= 6x^2 + 7x + 2$$

- Multiplying the first term in each bracket gives the '$6x^2$' term
- Multiplying the last term in each bracket gives the constant term (+2),
- The coefficient of the 'x' term is the sum of the 'x' terms: ($+3x + 4x = +7x$).

Factorisation of $ax^2 + bx + c$ with $a \neq 1$

Technique similar to those used for expressions of the type $x^2 + bx + c$ are suitable but in this case the coefficient of x , in at least one bracket, will not equal 1.

Examples

1. *Factorise $2x^2 + 7x + 6$*

$$2x^2 + 7x + 6 = (2x\ldots\quad)(x\ldots\quad)$$
$$= (2x + 3)(x + 2)$$

[$2x \times x = 2x^2$ and we need to choose factors of + 6]
[middle term +ve, try all combinations of + 6 and +1 or +3 and +2]
[gives $3x + 4x = 7x$ for middle term]

2. *Factorise $2x^2 - x - 6$*

$$2x^2 - x - 6 = (2x\ldots\quad)(x\ldots\quad)$$
$$= (2x + 3)(x - 2)$$

[$2x \times x = 2x^2$ and we need to choose factors of - 6]
[middle term -ve, try all combinations of $\pm$ 6 and $\pm$1 or $\pm$3 and $\pm$2]
[gives $3x - 4x = -x$ for middle term]

3. *Factorise $3a^2 - 16a + 5$*

$$3a^2 - 16a + 5 = (3a\ldots\ )(a\ldots\quad)$$
$$= (a - 5)(3a - 1)$$

[$3a \times a = 3a^2$ and we need to choose factors of + 5]
[middle term -ve, so -5 and -1 are the correct factors]
[gives $-15a - a = 16a$ for middle term]

4. *Factorise $6y^2 + 16y - 6$*

$$6y^2 + 16y - 6 = 2(3y^2 + 8y - 3)$$
$$= 2(3y\ldots\quad)(y\ldots\quad)$$
$$= 2(3y - 1)(y + 3)$$

[Remove a common factor of 2]
[$3y \times y = 3y^2$ and we need to choose factors of -6]
[middle term -ve, try all combinations of $\pm$ 6 and $\pm$1 or $\pm$3 and $\pm$2]
[gives $+9y - y = 8y$ for middle term]

Exercise

Factorise (if possible) the following

(1) $5x^2 + 13x + 6$	(2) $2x^2 + x - 15$	(3) $3m^2 - m - 2$
(4) $3y^2 - 10y + 8$	(5) $2a^2 + 11a + 12$	(6) $6x^2 - 11x + 5$

Answers

(1) $(5x + 3)(x + 2)$	(2) $(2x - 5)(x + 3)$	(3) $(3m + 2)(m - 1)$
(4) $(3y - 4)(y - 2)$	(5) $(2a + 3)(a + 4)$	(6) $(6x - 5)(x - 1)$

FACTORISATION: COMPLETING THE SQUARE

A quadratic expression may be factorised by the method of "completing the square" even when the factors are not rational.

A **perfect square** is of the form $(x + b)^2 = x^2 + 2bx + b^2$

The first two terms of the right-hand side of the above equation are $(x^2 + 2bx)$. To make a perfect square or to **complete the square** on $x^2 + 2bx$, the third term, b^2 must be added.

Note that b^2 = (half the coefficient of x)^2.

Examples

1) *Complete the square on* $x^2 + 6x$

   $x^2 + 6x$ are the first two terms of the complete square $x^2 + 6x + 9$ [(half of 6)^2, $= 3^2 = 9$]

   $= (x + 3)^2$

2) *Complete the square on* $x^2 + 5x$

   $x^2 + 5x$ are the first two terms of the complete square $x^2 + 5x + \frac{25}{4}$ [(half of 5)^2, $= (\frac{5}{2})^2 = \frac{25}{4}$]

   $= (x + \frac{5}{2})^2$

3) *Complete the square on* $x^2 - 10x$

   $x^2 - 10x$ are the first two terms of the complete square $x^2 - 10x + 25$ [(half of 10)^2, $= 5^2 = 25$]

   $= (x - 5)^2$

Factorisation

Expressions of the form $x^2 + bx + c$ can sometimes be factorised by completing the square on the $x^2 + bx$ and rewriting

$$x^2 + bx + c = (x + \tfrac{b}{2})^2 - \left(\tfrac{b}{2}\right)^2 + c$$

$$= (x + \tfrac{b}{2})^2 - \left(\left(\tfrac{b}{2}\right)^2 - c\right)$$

$$= (x + \tfrac{b}{2})^2 - m^2 \quad \text{where } m^2 = \left(\tfrac{b}{2}\right)^2 - c$$

The DOTS rule may then be used to factorise the expression.

Examples

Factorise by the method of completing the square

1) $x^2 + 8x - 5 = (x^2 + 8x + 16) - 16 - 5$ [Halve the coefficient of x and square: (8/2) =4 and 4^2 =16] [Add and subtract 16].

   $= (x + 4)^2 - 21$

   $= (x + 4)^2 - \sqrt{21}^2$ [Write as the difference of two squares].

   $= (x + 4 + \sqrt{21})(x + 4 - \sqrt{21})$ [Factorise using 'DOTS']

2) $x^2 - 6x - 4 = (x^2 - 6x + 9) - 9 - 4$ [Halve the coefficient of x and square: (6/2) =3 and 3^2 =9] [Add and subtract 9].

   $= (x - 3)^2 - 13$

   $= (x - 3)^2 - \sqrt{13}^2$ [Write as the difference of two squares].

   $= (x - 3 + \sqrt{13})(x - 3 - \sqrt{13})$ [Factorise using 'DOTS']

If the coefficient of x^2 is not 1, remove the coefficient as a common factor and then factorise by completing the square.

3) $2x^2 - 10x + 2 = 2(x^2 - 5x + 1)$ [Halve the coefficient of x and square: $(5/2)^2 = 25/4$
[Add and subtract $(5/2)^2$].

$= 2(x^2 - 5x + (\frac{5}{2})^2 - (\frac{5}{2})^2 + 1)$

$= 2[(x - \frac{5}{2})^2 - \frac{25}{4} + 1]$ [Simplify]

$= 2[(x - \frac{5}{2})^2 - \frac{21}{4}]$

$= 2[(x - \frac{5}{2})^2 - \left(\frac{\sqrt{21}}{2}\right)^2]$ [Write as the difference of two squares].

$= 2[(x - \frac{5}{2} + \frac{\sqrt{21}}{2})\ (x - \frac{5}{2} - \frac{\sqrt{21}}{2})]$

Exercise

Factorise by completing the square (if possible).

(1) $x^2 + 6x – 5$ (2) $a^2 – 8a – 2$ (3) $m^2 + 2m + 10$ (4) $4x^2 + 8x – 20$

Answers

(1) $(x + 3 + \sqrt{14})(x + 3 - \sqrt{14})$ (2) $(a - 4 + \sqrt{18})(a - 4 - \sqrt{18})$

(3) no real factors (4) $4(x + 1 + \sqrt{6})(x + 1 - \sqrt{6})$

ALGEBRAIC FRACTIONS

Simplifying fractions

Remember $\frac{18}{24} = \frac{\cancel{18}^3}{\cancel{24}^4} = \frac{3}{4}$ because 18 and 24 have a common factor of 6.

and $\frac{5}{20} = \frac{\cancel{5}^1}{\cancel{20}^4}$ because 5 and 20 have a common factor of 5

Algebraic fractions may be simplified in a similar way by cancelling factors that are common to the numerator and denominator.

Examples

1. $\frac{8x^2y}{6y^2} = \frac{\cancel{8}^4 xx\cancel{y}}{\cancel{6}^3 y\cancel{y}} = \frac{4x^2}{3y}$

2. $\frac{a(b+2c)}{2ab} = \frac{\cancel{a}(b+2c)}{2\cancel{a}b} = \frac{b+2c}{2b}$, $a \neq 0$ [when a variable is cancelled out of the denominator the answer is given with the condition that the variable must not be zero]

3. $\frac{m-n}{(m-n)^2} = \frac{\cancel{m-n}}{\cancel{(m-n)}(m-n)} = \frac{1}{(m-n)}$, $m - n \neq 0$ [m-n can be cancelled providing $m \neq n$]

4. $\frac{3x^2y}{6x+9y} = \frac{\cancel{3}x^2y}{\cancel{3}(2x+3y)} = \frac{x^2y}{2x+3y}$ [Factorising helps you to see factors!]

5. $\frac{p-2}{6p-3p^2} = \frac{p-2}{3p(2-p)} = \frac{p-2}{-3p(p-2)} = -\frac{1}{3p}$, $p - 2 \neq 0$ [it is **VERY** useful to know that $2 - p = -(p - 2)$!]

> NB: Only factors may be cancelled
>
> $\frac{x+2}{2y} \neq \frac{x+1}{y}$ because 2 is NOT a factor of $x + 2$

Exercise

Simplify the following fractions

1. $\frac{12\,ab^2}{8bc}$

2. $\frac{9u-18}{2u-4}$

3. $\frac{6t-9}{12-8t}$ HINT: $3 - 2t = -(2t - 3)$

4. $\frac{9r^2-3r}{16r-48r^2}$

Answers

1. $\frac{3ab}{2c}$ 2. $\frac{9}{2}$, $u - 2 \neq 0$ 3. $-\frac{3}{4}$, $2t - 3 \neq 0$ 4. $-\frac{3}{16}$, $r \neq 0$, $3r - 1 \neq 0$

Multiplication of Fractions

Remember $\frac{15}{8} \times \frac{24}{35} = \frac{\not{15}^3}{\not{8}^1} \times \frac{\not{24}^3}{\not{35}^7} = \frac{9}{7}$ [Any factor in the numerator can be cancelled with any factor in the denominator]

Similarly $\frac{5a}{7} \times \frac{14}{a} = \frac{5\not{a}^3}{\not{7}^1} \times \frac{\not{14}^2}{\not{a}^1} = 10, \; a \neq 0$

Examples

1. $\frac{x}{6(x-2)} \times \frac{3(x-2)}{x^2} = \frac{1}{2x}$, $x - 2 \neq 0$

2. $\frac{3m+12}{10} \times \frac{5}{m^2+4m} = \frac{3(m+4)}{10} \times \frac{5}{m(m+4)} = \frac{3}{2m}$, $m + 4 \neq 0$ [factorise before simplifying]

Division of Fractions

Remember $\frac{5}{12} \div \frac{19}{8} = \frac{5}{12} \times \frac{8}{19}$ [change to multiply and invert the second fraction]

$= \frac{5}{\not{12}^3} \times \frac{\not{8}^2}{19}$

$= \frac{10}{57}$

Division with algebraic fractions is very similar

- Invert and multiply
- Factorise (if necessary) and cancel
- Simplify

Examples

1. $\frac{7P}{12} \div \frac{3}{8} = \frac{7P}{12} \times \frac{8}{3}$

$= \frac{7P}{\not{12}^3} \times \frac{\not{8}^2}{3}$

$= \frac{14P}{9}$

2. $\frac{m^2}{n} \div 6m = \frac{m^2}{n} \div \frac{6m}{1}$

$= \frac{m^2}{n} \times \frac{1}{6m}$

$= \frac{m \times m}{n} \times \frac{1}{6m}$

$= \frac{m}{6n}$, $m \neq 0$

3. $\frac{2a+4}{15} \div \frac{a+2}{6} = \frac{2a+4}{15} \times \frac{6}{a+2}$

$= \frac{2(a+2)}{15} \times \frac{6}{a+2} = \frac{4}{5}$, $a + 2 \neq 0$

Exercise

Simplify

1. $\frac{32h^2}{9j} \times \frac{27j}{48h}$ 2. $\frac{3-2y}{33y-11} \times \frac{18y^2-6y}{7-2y}$ 3. $\frac{6xy-5y^2}{4x+10y} \div \frac{12x^2-10xy}{12x+30y}$

Answers

1. 2h h, j ≠ 0 2. $\frac{6y(3-2y)}{11(7-2y)}$, 3y – 1 ≠ 0 3. $\frac{3y}{2x}$, 2x + 5y ≠ 0, 6x – 5y ≠ 0

<u>Addition and Subtraction</u>

> **Only fractions which have a *common denominator* may be added or subtracted.**

Remember $\frac{7}{10} - \frac{3}{7} = \frac{7}{10}\cdot\frac{7}{7} - \frac{3}{7}\cdot\frac{10}{10}$ [70 is a common denominator as 7 and 10 are factors of 70]

$= \frac{49}{70} - \frac{30}{70}$ [Equivalent fractions are found with denominators of 70]

$= \frac{49-30}{70}$

$= \frac{19}{70}$

The process with algebraic fractions is very similar

Examples

1. $\frac{h}{6} + \frac{2h}{9} = \frac{h}{6}\cdot\frac{3}{3} + \frac{2h}{9}\cdot\frac{2}{2}$ [18 is a common denominator]

$= \frac{3h}{18} + \frac{4h}{18}$

$= \frac{3h+4h}{18}$

$= \frac{7h}{18}$

2. $\frac{e+1}{2} + \frac{e}{5} = \frac{e+1}{2} \cdot \frac{5}{5} + \frac{e}{5} \cdot \frac{2}{2}$ [10 is a common denominator]

$$= \frac{5(e+1)}{10} + \frac{2e}{10}$$

$$= \frac{5e+5+2e}{10}$$

$$= \frac{7e+5}{10}$$

3. $\frac{5}{2a} - \frac{3}{4} = \frac{5}{2a} \cdot \frac{2}{2} - \frac{3}{4} \cdot \frac{a}{a}$ ['4a' is a common denominator as 2, 'a' and 4 and are factors of 4a]

$$= \frac{10}{4a} - \frac{3a}{4a}$$

$$= \frac{10-3a}{4a}$$

Exercise

Simplify:

1. $\frac{4}{5}+\frac{3}{4}$ **2.** $\frac{x}{3}-\frac{x}{5}$ **3.** $\frac{2p}{7}-\frac{p}{4}$

4. $\frac{2g}{3}+\frac{g+1}{4}$ **5.** $\frac{d+3}{2}+\frac{1-d}{4}$ **6.** $\frac{5}{9}-\frac{3}{b}$

7. $\frac{3x+2}{5} - \frac{x-3}{10}$ **8.** $\frac{3}{v}+\frac{2}{v+1}$

Answers

1 $\frac{31}{20}$ 2 $\frac{2x}{15}$ 3 $\frac{p}{28}$ 4 $\frac{11g+3}{12}$

5 $\frac{d+7}{4}$ 6 $\frac{5b-27}{9b}$ 7 $\frac{5x+7}{10}$ 8 $\frac{5v+3}{v(v+1)}$

Complicated Fractions

More complicated algebraic fractions may contain quadratic expressions or fractions within fractions.

Examples:

1. $\dfrac{2}{x^2+x-12}-\dfrac{1}{x^2-9} = \dfrac{2}{(x+4)(x-3)} - \dfrac{1}{(x-3)(x+3)}$ [factorize to see the factors in the denominator]

$$= \frac{2(x+3)}{(x+4)(x-3)(x+3)} - \frac{(x+4)}{(x+4)(x-3)(x+3)}$$

[we must have a common denominator before we add fractions and the common denominator must contain one of every factor]

$$= \frac{2(x+3) - (x+4)}{(x+4)(x-3)(x+3)}$$

$$= \frac{x+2}{(x+4)(x-3)(x+3)}$$

2. Simplify $\dfrac{1-\frac{2}{x}}{1+\frac{2}{x}}$

$$\frac{1-\frac{2}{x}}{1+\frac{2}{x}} = \left(1 - \frac{2}{x}\right) \div \left(1 + \frac{2}{x}\right)$$

$$= \left(\frac{x}{x} - \frac{2}{x}\right) \div \left(\frac{x}{x} + \frac{2}{x}\right)$$

$$= \frac{x-2}{x} \div \frac{x+2}{x}$$

$$= \frac{x-2}{x} \times \frac{x}{x+2}$$

$$= \frac{x-2}{x+2}, \qquad x \neq 0$$

3. Simplify $\dfrac{\dfrac{-3}{x^2+2x-3}+\dfrac{1}{x-1}}{\dfrac{3}{x-1}+3}$

$$\frac{\frac{-3}{x^2+2x-3}+\frac{1}{x-1}}{\frac{3}{x-1}+3} = \left(\frac{-3}{x^2+2x-3}+\frac{1}{x-1}\right) \div \left(\frac{3}{x-1}+3\right)$$

$$= \left(\frac{-3}{(x+3)(x-1)}+\frac{x+3}{(x+3)(x-1)}\right) \div \left(\frac{3}{x-1}+\frac{3(x-1)}{x-1}\right)$$

$$= \frac{-3+x+3}{(x+3)(x-1)} \div \frac{3+3x-3}{x-1}$$

$$= \frac{x}{(x+3)(x-1)} \div \frac{3x}{x-1}$$

$$= \frac{x}{(x+3)(x-1)} \times \frac{x-1}{3x}$$

$$= \frac{1}{3(x+3)} \quad x \neq 1,\ x \neq 0$$

Exercise

1. $\dfrac{2}{x^2+2x}+\dfrac{1}{x^2-4}$

2. $\dfrac{1-\frac{6}{x}}{\frac{x}{2}-3}$

3. $\dfrac{\frac{1}{x^2-4}+\frac{1}{2x+4}}{1+\frac{2}{x-2}}$

Answers

1. $\dfrac{3x-4}{x(x+2)(x-2)}$ 2. $\dfrac{2}{x},\ x \neq 6$ 3. $\dfrac{1}{2(x+2)},\ x \neq 0,\ x \neq 2$

PARTIAL FRACTIONS

Adding fractions

To add fractions, rewrite the fractions with a common denominator then add the numerators.

For example:

$$\frac{3}{3x+4}+\frac{2}{x-5} = \frac{3}{3x+4}\times\frac{x-5}{x-5}+\frac{2}{x-5}\times\frac{3x+4}{3x+4}$$ [a common denominator is $(3x+4)(x-5)$]

$$= \frac{3x-15}{(3x+4)(x-5)}+\frac{6x+8}{(3x+4)(x-5)}$$

$$= \frac{3x-15+6x+8}{(3x+4)(x-5)}$$

$$= \frac{9x-7}{(3x+4)(x-5)}$$

Finding Partial fractions

The reverse of this process is to split a fraction into partial fractions. In the above example

$$\frac{9x-7}{(3x+4)(x-5)} = \frac{3}{3x+4}+\frac{2}{x-5}$$

Algebraic fraction

Partial fractions

The first step in finding partial fractions is to factorise the denominator. The factorisation will determine the form of the partial fractions:

$\frac{mx+k}{(x-a)(x-b)}$	(distinct linear factors)	$\frac{A}{x-a}+\frac{B}{x-b}$	A, B constant
$\frac{mx+k}{(x-a)^2}$	(repeated linear factor)	$\frac{A}{x-a}+\frac{B}{(x-a)^2}$	A, B constant
$\frac{mx+k}{(ax^2+bx+c)(px+q)}$	(quadratic factor and linear factor)	$\frac{A}{px+q}+\frac{Bx+C}{ax^2+bx+c}$,	A, B, C constant

Exercise

Rewrite each of the following in the appropriate generalized (do not calculate the constants) partial fractions form.

a) $\frac{x+6}{2x^2+5x-12}$ b) $\frac{2x}{(x^2+3)(x+1)}$ c) $\frac{2x}{x^2+8x+16}$ d) $\frac{x^2}{(x-1)(x+1)}$

Answers

a) $\frac{A}{2x-3}+\frac{B}{x+4}$ b) $\frac{Ax+B}{x^2+3}+\frac{C}{x+1}$ c) $\frac{A}{x+4}+\frac{B}{(x+4)^2}$ d) $1+\frac{A}{x+1}+\frac{B}{x-1}$

To express an algebraic fraction as partial fractions:

1. Factorise the denominator.

2. Write the algebraic fraction in partial fraction form with unknown constants as above

3. Add the partial fractions

4. Then equate coefficients , or substitute values of x, to determine the value of the constants

Examples

1. Express $\frac{x-5}{x^2+2x-3}$ as the sum of partial fractions

$\frac{x-5}{x^2+2x-3} = \frac{x-5}{(x-1)(x+3)} \Rightarrow \frac{x-5}{x^2+2x-3} = \frac{A}{x-a} + \frac{B}{x-b}$ [distinct linear factors]

$\Rightarrow \frac{x-5}{x^2+2x-3} = \frac{A}{x-1} + \frac{B}{x+3}$

$\Rightarrow \frac{x-5}{x^2+2x-3} = \frac{A(x+3)+B(x-1)}{x^2+2x-3}$ [common denominator is $(x-1)(x+3)$]

$\Rightarrow \quad x-5 = A(x+3) + B(x-1)$ [Equating numerators]

$\Rightarrow \quad x-5 = Ax + 3A + Bx - B$ [Expanding the brackets]

$\Rightarrow \quad x-5 = (A+B)x + (3A-B)$ [Collect like terms of x]

Equating coefficients of x we find that

$A + B = 1$ and

$3A - B = -5$

Solving these simultaneous equations we find the $A = -1$ and $B = 2$

therefore $\frac{x-5}{x^2+2x-3} = \frac{-1}{x-1} + \frac{2}{x+3}$

Alternative method

Solving the simultaneous equations that result from equating coefficients can sometimes be quite lengthy. An alternative method is to equate the numerators and, before expanding the brackets, substitute a value of x into both sides of the equation so that only one variable remains. Repeat this to find other variables. This method will not necessarily find all variables, but will often make calculations easier.

In the above example, after the line: $x-5 = A(x+3) + B(x-1)$ we can substitute any convenient value for x

$$x-5 = A(x+3) + B(x-1)$$

If $x = -3$ then $-3 - 5 = A(0) + B(-3 - 1)$ $\Rightarrow -8 = -4B$

$\Rightarrow B = 2$

If $x = 1$ then $1 - 5 = A(1 + 3)) + B(0)$ $\Rightarrow -4 = 4A$

$\Rightarrow A = -1$

Using the method of substituting convenient values for x

$\therefore \quad \frac{x-5}{x^2+2x-3} = \frac{-1}{x-1} + \frac{2}{x+3}$

This is the same answer as we have above, using simultaneous equations to solve for *A* and *B.*

2. Express $\frac{5x^2+3x+1}{x^3-3x-2}$ as the sum of partial fractions

$\frac{5x^2+3x+1}{x^3-3x-2} = \frac{5x^2+3x+1}{(x+1)^2(x-2)}$ [factorising the denominator]

$\Rightarrow \quad \frac{5x^2+3x+1}{(x+1)^2(x-2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-2}$ [repeated linear factors]

$\Rightarrow \quad \frac{5x^2+3x+1}{(x+1)^2(x-2)} = \frac{A(x+1)(x-2)+B(x-2)+C(x+1)^2}{(x+1)^2(x-2)}$ [finding a common denominator]

$\Rightarrow \quad 5x^2 + 3x + 1 = A(x+1)(x-2) + B(x-2) + C(x+1)^2$ [Equating numerators]

We can now substitute any convenient value for *x*

If $x = 2$ then $5(2)^2 + 3(2) + 1 = A(0) + B(0) + C(3)^2$

$\Rightarrow \quad 27 = 9C$

$\Rightarrow \quad C = 3$

If $x = -1$ then $5(-1)^2 + 3(-1) + 1 = A(0) + B(-1-2) + C(0)^2$

$\Rightarrow \quad 3 = -3B$

$\Rightarrow \quad B = -1$

If $x = 0$ then $5(0)^2 + 3(0) + 1 = A(1)(-2) + -1(-2) + 3(1)^2$ [using $B = -1$ and $C = 3$]

$\Rightarrow \quad 1 = -2A + 2 + 3$

$\Rightarrow \quad A = 2$

Therefore $\frac{5x^2+3x+1}{(x+1)^2(x-2)} = \frac{2}{x+1} - \frac{1}{(x+1)^2} + \frac{3}{x-2}$

3. Express $\frac{x^2-3x+16}{x^3-5x^2+x-5}$ as the sum of partial fractions

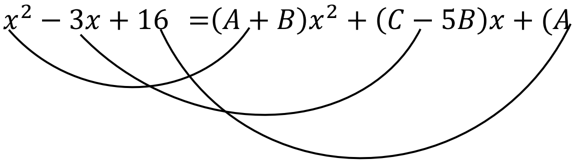
$\frac{x^2-3x+16}{x^3-5x^2+x-5} = \frac{x^2-3x+16}{(x^2+1)(x-5)}$

$\frac{x^2-3x+16}{x^3-5x^2+x-5} = \frac{A}{x-5} + \frac{Bx+C}{x^2+1}$ [linear and quadratic factors]

$\frac{x^2-3x+16}{x^3-5x^2+x-5} = \frac{A(x^2+1)+(Bx+C)(x-5)}{x^3-5x^2+x-5}$ [common denominator is $x^3 - 5x^2 + x - 5$]

$x^2 - 3x + 16 = A(x^2+1) + (Bx+C)(x-5)$ [equating numerators]

$x^2 - 3x + 16 = (A+B)x^2 + (C-5B)x + (A-5C)$ [removing brackets and regrouping]



$\therefore \quad A + B = 1$

$C - 5B = -3$ — Using the method of equating coefficients

$A - 5C = 16$

and $\quad A = 1$

$B = 0$ — by solving simultaneous equations

$C = -3$

Therefore $\frac{x^2-3x+16}{x^3-5x^2+x-5} = \frac{1}{x-5} - \frac{3}{x^2+1}$

The process of finding partial fractions can only be performed on fractions where the degree of the numerator of the algebraic fraction is greater than that of the denominator. If necessary, divide the denominator into the numerator then express the remaining fractional part as partial fractions.

For example $\frac{x^2+7x+7}{(x+1)(x+2)} = 1 + \frac{4x+5}{(x+1)(x+2)} = 1 + \frac{1}{x+1} + \frac{3}{x+2}$ [after expressing as partial fractions]

Exercise

Express the following as partial fractions

a) $\frac{x+2}{x^2-5x+6}$ b) $\frac{3}{x^2-2x+1}$ c) $\frac{x^2-2x+2}{x^3+x^2+x}$ d) $\frac{x^2}{x^2-4}$

Answers

a) $\frac{5}{x-3} - \frac{4}{x-2}$ b) $\frac{3}{(x-1)^2}$ c) $\frac{2}{x} - \frac{x+4}{x^2+x+1}$ d) $1 - \frac{1}{x+2} + \frac{1}{x-2}$