Algebra Skills
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ALGEBRAIC OPERATIONS

Like Terms

Like terms contain exactly the same pronumerals (letters, variables)

<table>
<thead>
<tr>
<th>Like Terms</th>
<th>Unlike terms</th>
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<tbody>
<tr>
<td>3x, 5x</td>
<td>3x, 4y</td>
</tr>
<tr>
<td>2a, -3a</td>
<td>3a, 3</td>
</tr>
<tr>
<td>3m², m²</td>
<td>3m², 3m</td>
</tr>
<tr>
<td>2ab, 3ab</td>
<td>2a²b, 3ab²</td>
</tr>
<tr>
<td>3ef, 5fe</td>
<td>3ef, 7fg</td>
</tr>
</tbody>
</table>

NB: Order is unimportant but alphabetical order of pronumerals is conventional

See Exercise 1

Addition and Subtraction

Only like terms may be added or subtracted.

1. \(7e + 10e = (7 + 10)e\) \[eventually\ it\ is\ possible\ to\ just\ THINK\ the\ second\ step\ and\ go\ straight\ to\ the\ answer\]
   \[= 17e\]

2. \(3x^2 - x^2 - 4x^2 = (3 - 1 - 4)x^2\)
   \[= -2x^2\]

3. \(3m - 4n + 6m + n = (3 + 6)m + (-4 + 1)n\)
   \[= 9m - 3n\]

4. \(3a - b - 5a + 4ab - 3b + ab = (3 - 5)a + (-1 - 3)b + (4 + 1)ab\)
   \[= -2a - 4b + 5ab\]

5. \(3x - x^2\) cannot be simplified

6. \(p + 2p - 3 = 3p - 3\)

7. \(8uv + 3u - 10vu = -2uv + 3u\)

8. \(6rx^2 - 2rs^2\) cannot be simplified

See Exercise 2

Multiplication

Consider the sign (positive or negative) of the answers to the following simple multiplication problems

\[2 \times 3 = 6 \quad \text{positive x positive} \rightarrow \text{positive}\]

\[-2 \times 3 = -6 \quad \text{negative x positive} \rightarrow \text{negative}\]

\[2 \times -3 = -6 \quad \text{positive x negative} \rightarrow \text{negative}\]

\[-2 \times -3 = 6 \quad \text{negative x negative} \rightarrow \text{positive}\]

Terms can be multiplied whether they are like or unlike terms.
When multiplying two or more terms consider
- the sign of the answer
- the product of the numbers

and use an index to show how many factors of each pronumeral

1. \((-4) \times (-3b) = 12b\)
2. \(-2 \times 6y = -12y\)
3. \(2e \times (-5e^2) = -10e^3\) [three factors of ‘e’ altogether]
4. \((-2u^2v) \times (-4v) = 8u^2 v^2\)
5. \(-3pq \times (-2q) \times p = 6p^2 q^2\)

In algebra a fraction line means to divide eg. \(\frac{1}{2} = 1 \div 2\), \(\frac{3}{4} = 3 \div 4\)

When dividing algebraic expressions
- rewrite as a fraction if necessary
- expand any powers
- establish the sign of the answer
- cancel factors
- write the answer in simplified form

1. \(-12wyz ÷ 3yz = \frac{-12wyz}{3yz} = -4w\)
2. \(-2m^2n ÷ 6mn^2 = \frac{-2m^2n}{6mn^2} = \frac{-2mn}{6mn} = -\frac{m}{3n}\)
3. \(6a^2 \times 4b ÷ 12ab = \frac{6aa \times 4b}{12ab} = 2a\)

\[\text{NB:} \quad \frac{m+2}{4m} \text{ CANNOT be simplified!!!}\]

See Exercise 3

Order of Operations
Consider \(4 + 3 \times 2 + 5\). What is the answer?

\[
4 + 3 \times 2 + 5 \rightarrow 7 \times 7 = 49? \\
4 + 3 \times 2 + 5 \rightarrow 4 + 3 \times 7 = 25? \\
4 + 3 \times 2 + 5 \rightarrow 7 \times 2 + 5 = 19?
\]

To avoid such confusion we perform operations in the following order...
1. **Brackets**  

2. **Indices**

3. **Multiplication and Division from left to right**

4. **Addition and Subtraction from left to right**

To make it easier to remember the rule for order of operations is abbreviated to **B I M D A S**

Using this rule  \[ 4 + 3 \times 2 + 5 = 4 + 6 + 5 = 15!! \]

**Examples:**

1. \[ 3 \times 2 + 4 = 6 + 4 = 10 \]
2. \[ 3 + 2 \times 4 = 3 + 8 = 11 \]
3. \[ (3 + 2) \times 4 = 5 \times 4 = 20 \]
4. \[ 3 - 2^2 = 3 - 4 = -1 \]
5. \[ (3 - 2)^2 = 1^2 = 1 \]
6. \[ 3^2 - 2^2 = 3 \times 3 - 2 \times 2 = 9 - 4 = 5 \]
7. \[ 3st - 3s \times 4t = 3st - 12st = -9st \]

**See Exercise 4**

**Exercises**

**Exercise 1**

Which of the five terms on the right is a like term with the term on the left?

1) \[ 3x \quad 3 \quad 2x \quad 3x^2 \quad 2xy \quad 4x^2a \]
2) \[ 2ab \quad 2a \quad 2b \quad 3x^2 \quad 6abc \quad 12ab \]
3) \[ 2x^2 \quad x \quad 2x \quad 5x^2 \quad 4x \quad 4x^3 \]
4) \[ 3xy^2 \quad 3 \quad 3xy \quad 3x^2 \quad xy^2 \quad 3y^2 \]
5) \[ 2m^2n \quad n \quad mn^2 \quad 8mn \quad 2m^2 \quad 4nm^2 \]
6) \[ 4ab^2c \quad 4 \quad 2ab^2c \quad 4abc \quad 8b^2c \quad 4cba \]

**Exercise 2**

Simplify each of the following.

1. a) \[ 5x + 3x \]
   c) \[ 15xy + 5xy \]
   e) \[ 10abc - 3bca \]
   g) \[ 6x + 3x + 4x \]
   i) \[ m + 2m - 9m \]
   
   b) \[ 12x - 7x \]
   d) \[ 11mn - 5mn \]
   f) \[ 10m - 22m \]
   h) \[ 4ab + 5ab - 2ab \]
   j) \[ 14xy - 4xy + 2xy \]

2. a) \[ 13x + 4 - 3x -1 \]
   c) \[ 3xy^2 + 2xy + 5xy^2 + 3xy \]
   e) \[ x + y + 2x -y \]
   g) \[ x + 4y - x - 2y \]
   
   b) \[ 10mn + 5m + 12mn + 6m \]
   d) \[ 5xy + 6m - 2xy -2m \]
   f) \[ 3a + 5b - a - 6b \]
   h) \[ 7x - 4m - 5x - 3m \]
Exercise 3
Simplify the following algebraic expressions

1. a) \(5 \times 2k\)  
   b) \(4a \times 3ab\)  
   c) \(y \times 3y\)  
   d) \(4m \times (-3mn)\)  
   e) \(m \times 3p \times 5\)  
   f) \(2ab \times 3bc \times (-4)\)  
   g) \(2ab^2 \times 3ac\)  
   h) \(4m \times (-5km)\)

2. a) \(18ef \div 6f\)  
   b) \(-100uvw \div 100w\)  
   c) \(24gh^2 \div 8gh\)  
   d) \(3m^2 n \div 12mn^2\)  
   e) \(rs \times 2st \div 2s\)  
   f) \(3jk \times 12km \div 9jkm\)  
   g) \(10p \times 3pq \div 16pq\)  
   h) \(4yz \times 5w^2z \div 10wy\)

Exercise 4
Simplify

1) \(18 + (3 \times -5)\)  
2) \(3 \times -4 + (8 \times 2)\)  
3) \(10 - 5^2 + 3\)  
4) \((10 - 5)^2 + 3\)  
5) \(10^2 - 5^2\)  
6) \((10 - 5)^2\)  
7) \(3m + 2 \times 3m\)  
8) \(6ab - 3a \times 4b\)  
9) \(16gh - 4gh \times 4\)  
10) \(9b - 3b \times 2k + 2k \times b\)

Answers
Exercise 1
1. 2x  
2. 12ab  
3. 5x^2  
4. xy^2  
5. 4nm^2  
6. 2ab^2c

Exercise 2
1. a. 6x  
   b. 5x  
   c. 20xy  
   d. 6mn  
   e. 7abc  
   f. -12m  
   g. 13x  
   h. 7ab  
   i. -6m  
   j. 12xy
2. a. 10x + 3  
   b. 22mn + 11m  
   c. 8xy^2 + 5xy  
   d. 3xy + 4m  
   e. 3x  
   f. 2a - b  
   g. 2y  
   h. 2x - 7m  
   i. 4x - 3y  
   j. 9mn - 3m - n + 4m^2

Exercise 3
1. a. 10k  
   b. 12a^2b  
   c. 3y^2  
   d. -12m^2n  
   e. 15mp  
   f. -24ab^2c  
   g. 6a^2b^2c  
   h. -20km^3p

2. a. 3e  
   b. -uv  
   c. 3h  
   d. \(\frac{m}{4n}\)  
   e. rst  
   f. 4k  
   g. \(\frac{15p}{8}\)  
   h. 2wz^2

Exercise 4
1. 3  
2. 4  
3. -12  
4. 28  
5. 75  
6. 25  
7. 9m  
8. -6ab  
9. 0  
10. 9b - 4bk
ALGEBRAIC SUBSTITUTION

The process of replacing a pronumeral (or variable or letter) in an expression or formula with a number is called substitution.

After substitution it is possible to evaluate the algebraic expression.

Examples

1. Evaluate $3p$ if $p = -5$

   
   
   $3p = 3 \times p$
   
   $= 3 \times -5$
   
   $= -15$

2. Evaluate $\frac{a+5}{b}$ if $a = -9$ and $b = 2$

   
   
   $\frac{a+5}{b} = \frac{-9+5}{2}$
   
   $= \frac{-4}{2}$
   
   $= -2$

3. Evaluate $w^2 - 2z$ if $w = -1$, $z = 5$

   
   
   $w^2 - 2z = (-1)^2 - 2 \times 5$
   
   $= 1 - 10$
   
   $= -9$

4. Use the formula $C = \frac{5(F-32)}{9}$ to convert a temperature of 100° in the Fahrenheit system to Centigrade.

   
   
   $C = \frac{5(F-32)}{9}$
   
   $= \frac{5(100-32)}{9}$
   
   $= \frac{5(68)}{9}$
   
   $= 37.78$
Exercises

1. Evaluate the following

   a) \(-4k\) if \(k = 7\) 
   b) \(2mn\) if \(m = 4, n = -2\) 
   c) \(e^2 - 5\) if \(e = 2\) 
   d) \(5 - b + b^2\) if \(b = 3\) 
   e) \(2k^2 + 4\) if \(k = -6\) 
   f) \(-3ab^2\) if \(a = 4, b = 2\) 
   g) \(\frac{n}{4} + 2\) if \(n = 10\) 
   h) \(\frac{u}{5v}\) if \(u = -20, v = 2\)

2. Evaluate the following if \(a = -1, b = 6, c = 3, m = -2, n = 2\)

   a) \(3a^2 - 7\) 
   b) \(3a - b^2\) 
   c) \((2m + 1)^2\) 
   d) \(3(a^2 + 4)\) 
   e) \(\frac{2m}{n}\) 
   f) \((m - n)^2\)

Answers

1. a) -28 b) -16 c) -1 d) 11 e) 76 f) -48 g) 4.5 h) -2
2. a) -4 b) -39 c) 9 d) 15 e) -2 f) 16
REMOVING BRACKETS

To expand brackets like \( a(b + c) \) ie \( a \times (b + c) \) we use the rule:

\[
a \times (b + c) = a \times b + a \times c
\]
or \( a(b + c) = ab + ac \)

so \( 2 \times (3 + 5) = 2 \times 3 + 2 \times 5 \)

Examples

1. \( 3(x + 2) = 3 \times x + 3 \times 2 \)
   \( = 3x + 6 \)

2. \( -5(m + 4) = (-5) \times m + (-5) \times 4 \)
   \( = -5m + (-20) \)
   \( = -5m - 20 \)

3. \( -2(p - 7) = -2p + 14 \)

4. \( e(e + 2) = e^2 + 2e \) [leaving out the second step]

5. \( -(4p - 3) = (-1)(4p - 3) \) [nb: \( x = (-1)x \)]
   \( = -4p + 3 \)

6. \( 4(x - 3) + 2(x + 1) = 4x - 12 + 2x + 2 \)
   \( = 6x - 10 \) [after simplifying]

7. \( 6k(2k + 3) - 3k(k - 3) = 12k^2 + 18k - 3k^2 + 9k \)
   \( = 9k^2 + 27k \)

See Exercise 1

Binomial Products

A binomial product looks like this: \((m + 2) (m + 3)\).

It is the product of two binomial (two term) expressions. To expand the two brackets, multiply each term in one bracket by each term in the other bracket and simplify if possible.

\[
(m + 2)(m + 3) = m(m + 3) + 2(m + 3)
\]
\[
= m^2 + 3m + 2m + 6
\]
\[
= m^2 + 5m + 6
\]

These steps can be shown algebraically as:

\[
(A + B)(C + D) = AC + AD + BC + BD
\]

Important special cases of the expansion are:

\[
(a + b)^2 = (a + b)(a + b)
\]
\[
= a^2 + 2ab + b^2
\]

\[
(a - b)^2 = (a - b)(a - b)
\]
\[
= a^2 - 2ab + b^2
\]
Examples

1. \((2f + 3)(g + 5h) = 2f \times g + 3 \times g + 2f \times 5h + 3 \times 5h\)
   \[= 2fg + 3g + 10fh + 15h\]  
   [sometimes it is not possible to simplify further]

2. \((p - 2)(p - 7) = p^2 - 2p - 7p + 14\)
   \[= p^2 - 9p + 14\]

3. \((2w + 3)(w - 4) = 2w^2 + 3w - 8w - 12\)
   \[= 2w^2 - 5w - 12\]

4. \((4r - 3s)^2 = (4r - 3s)(4r - 3s)\)
   \[= 16r^2 - 12rs - 12rs + 9s^2\]
   \[= 16r^2 - 24rs + 9s^2\]

See Exercise 2

Exercises

Exercise 1

1 Remove the brackets
   a) \(2(x - 5)\)  b) \(3(6 - b)\)  c) \((b + 5)c\)
   d) \(2x(2a + 3b)\)  e) \(-6(p - 2q + 4)\)  f) \(3y(yz - 2y + 1)\)

2 Expand and simplify
   a) \(3(x + 1) + 2(x + 2)\)  b) \(2(p - 1) - (p - 3)\)
   c) \(2(4m - 3) - 5(9 - 2m)\)  d) \(2w(3w + 1) + 3w(2w - 5)\)

3 Expand the following expressions and simplify where possible remembering the correct order of operations
   a) \(4(2m - 3) + 8\)  b) \(9 - 3(4b + 3)\)
   c) \(1 - 4(x - 1)\)  d) \(8m - 3(1 - 2m) + 6\)

Exercise 2

1 Expand the following binomial products
   a) \((x + 5)(x + 3)\)  b) \((u - 5)(u + 3)\)
   c) \((2c - d)(3c + 2d)\)  d) \((3a - 2b)(3a + 2b)\)

2 Remove the brackets
   a) \((d + 3)^2\)  b) \((4f - 3gh)^2\)

Answers

Exercise 1

1 a) \(2x - 10\)  b) \(18 - 3b\)  c) \(bc + 5c\)  d) \(4ax + 6bx\)
   e) \(-6p + 12q - 24\)  f) \(3y^2z - 6y^2 + 3y\)

2 a) \(5x + 7\)  b) \(p + 1\)  c) \(18m - 51\)  d) \(12w^2 - 13w\)

3 a) \(8m - 4\)  b) \(-12b\)  c) \(5 - 4x\)  d) \(14m + 3\)

Exercise 2

1 a) \(x^2 + 8x + 15\)  b) \(u^2 - 2a - 15\)  c) \(6c^2 + cd - 2d^2\)
   d) \(9a^2 - 4b^2\)

2 a) \(d^2 + 6d + 9\)  b) \(16f^2 - 24gh + 9g^2h^2\)
FACTORISATION: COMMON FACTORS

Expansion of brackets (or removing brackets) in an algebraic expression is done by multiplying all the terms inside the brackets by the term(s) outside the brackets.

\[ 5a(3y - 2x) = 15ay - 10ax \]
Each term inside the brackets has been multiplied by 5a.

Factorisation is the reverse of expansion. To factorise a number or algebraic expression means to write the number or expression as a product (multiplication).

Examples
1. \(-2xyz\) has factors \(-2, x, y,\) and \(z\)
2. \(7(a + b)\) has factors 7 and \((a + b)\)
3. \(3(x - y)\) has factors 3 and \((x - y)\)
4. \(x(3a + 2b)\) has factors \(x\) and \((3a + 2b)\)
5. \(2p(2r + 1)(s + 4)\) has factors \(2, p, (2r + 1)\) and \((s + 4)\)

Factorisation by removing a common factor

- Search each term in the expression for a common factor (every term must have this factor)
- There may be several common factors. Search until you have found all of them
- If there is more than one common factor multiply them to give highest common factor. (HCF)
- The HCF is placed before the bracket
- The terms inside the bracket are found by dividing each term by the HCF.

Examples
1. \(5y + 10\)
   \[ 5y + 10 = 5y + 5 \times 2 \]
   \[ \text{common factor of 5} \]
   \[ 5 \times 2 = 10 \]
   \[ 5y + 10 = 5(y + 2) \]
   \[ \text{5 before brackets, \((y + 2)\) inside brackets} \]
2. \(3x + 3y\)
   \[ 3x + 3y = 3 \times x + 3 \times y \]
   \[ \text{common factor 3} \]
   \[ = 3(x + y) \]
Similarly
3. \( p^2 + p = p \times p + p \times 1 \) \hspace{1cm} \text{[common factor } p]\]
   \[= p(p + 1)\]

4. \( 7y^2 + 7y = 7y \times y + 7y \times 1 \) \hspace{1cm} \text{[common factors 7 and } y \text{, HCF=7]}
   \[= 7y(y + 1)\]

5. \( 2abc - 12ac = 2 \times a \times b \times c - 2 \times 6 \times a \times c \)
   \[= 2ac(b - 6) \] \hspace{1cm} \text{[common factors 2, a, and } c \text{, HCF = 2ac]}

\textbf{See Exercise 1}

A common factor may be negative.

\textbf{Examples}
1. \(-2a - 2b = (-2) \times a + (-2) \times b \) \hspace{1cm} \text{[common factor } -2]\]
   \[= -2(a + b)\]

2. \(-3x + 6xy = (-3x) \times 1 - (-3x) \times 2y \) \hspace{1cm} \text{[HCF = } -3x]\]
   \[= -3x(1 - 2y)\]

\textbf{See Exercise 2}

\textbf{Exercises}

\textbf{Exercise 1}

Factorise the following (if possible)

\begin{align*}
a & \text{) } 3x + 3y & b & \text{) } 8a - 8b + 8c & c & \text{) } xy - 5x & d & \text{) } 7x + 21y \\
e & \text{) } 5m - 2n & f & \text{) } 5mn - 10n & g & \text{) } 3m^2 - 3mn & h & \text{) } 12m^2 - 18mn \\
i & \text{) } 2ab^2c + 6abc^2 & j & \text{) } 5xyz - x^2yz^2 + 10x
\end{align*}

\textbf{Exercise 2}

Factorise the following by removing a negative factor.

\begin{align*}
a & \text{) } -3x - 6y & b & \text{) } -15xy + 25xz & c & \text{) } -6xyz - 15yz - 3xy^2z
\end{align*}

\textbf{Answers}

\textbf{Exercise 1}

\begin{align*}
a & \text{) } 3(x + y) & b & \text{) } 8(a - b + c) & c & \text{) } x(y - 5) & d & \text{) } 7(x + 3y) \\
e & \text{) } 2abc(b + 3c) & f & \text{) } 5n(m - 2) & g & \text{) } 3m(m - np) & h & \text{) } 6m(2m - 3n) \\
i & \text{) } x(5yz - xyz^2 + 10)
\end{align*}

\textbf{Exercise 2}

\begin{align*}
a & \text{) } -3(x + 2y) & b & \text{) } -5x(3y - 5z) & c & \text{) } -3yz(2x + 5 + xy)
\end{align*}
FACTORISATION: QUADRATICS (i)

Definition
The general form of a quadratic expression is

\[ ax^2 + bx + c, \quad a \neq 0 \]

where \( a, b, c \) are constants and \( x \) is the unknown

Review of Expansion
To expand an expression of the form \((a + b)(c + d)\), multiply each term in the first bracket by each term in the second bracket.

For example: \((x + 2)(x + 3) = x(x + 3) + 2(x + 3) = x^2 + 3x + 2x + 6\) [Simplify by combining like terms]

\[ = x^2 + 5x + 6 \]

Factorisation of expressions with \( a = 1 \)
If \( a = 1 \) the expression becomes \( x^2 + bx + c \)

Factorisation is the reverse of expansion.

\[ x^2 + 5x + 6 \] is expressed as the product of two factors, \((x + 2)\) and \((x + 3)\)

\[ \text{factorisation} \]

\[ x^2 + 5x + 6 = (x + 2)(x + 3) \]

\[ \text{expansion} \]

Note that:
- multiplying the first term in each bracket gives the term ‘\( x^2 \)’ in the expression
- multiplying the last term in each bracket gives the constant term, +6, in the expression
- the coefficient of the ‘\( x \)’ term is the sum of last term in each bracket (+2 + 3 = +5).

In general
- if the constant ‘\( c \)’ is positive
  \[ x^2 + bx + c = (x + m)(x + n) \text{ and } x^2 - bx + c = (x - m)(x - n) \]
- if the constant ‘\( c \)’ is negative
  \[ x^2 + bx - c = (x + m)(x - n), \quad m > n \text{ and } x^2 - bx - c = (x - m)(x + n) \quad m < n \text{ and } m, n \geq 0. \]

To factorise:

| Find two numbers \( m \) and \( n \) that: |
|------------------|------------------|
| • multiply to ‘\( c \)’ and |
| • add to ‘\( b \)’ |
| Insert into brackets \((x \ldots)(x \ldots)\) |

If ‘\( c \)’ is positive, both numbers are positive or both numbers are negative.
If ‘\( b \)’ is positive, both numbers are positive.
If ‘\( b \)’ is negative, both numbers are negative.

If ‘\( c \)’ is negative, then one number is positive and the other negative; if ‘\( b \)’ is positive, then the larger number is positive.
Order is not important. If b is negative, then the larger number is negative.

Examples

Factorise the following.

1. \( x^2 + 9x + 14 = (x\, \ldots\, ) (x\, \ldots\, ) \)
   \[= (x + 2)(x + 7) \]
   [the first term in each bracket must be 'x']
   [Find \( m \) and \( n \) so that \( m \times n = 14 \) and \( m + n = 9 \)
   \( \Rightarrow m = +2 \) and \( n = +7 \)]

2. \( y^2 - 7y + 12 = (y\, \ldots\, ) (y\, \ldots\, ) \)
   \[= (y - 3)(y - 4) \]
   [the first term in each bracket must be 'y']
   [Find \( m \) and \( n \) so that \( m \times n = 12 \) and \( m + n = -7 \)
   \( \Rightarrow m = -3 \) and \( n = -4 \)]

3. \( p^2 - 5p - 24 = (p\, \ldots\, ) (p\, \ldots\, ) \)
   \[= (p - 8)(p + 3) \]
   [the first term in each bracket must be 'p']
   [Find \( m \) and \( n \) so that \( m \times n = -24 \) and \( m + n = -5 \)
   \( \Rightarrow m = -8 \) and \( n = +3 \) since larger number must be -ve]

4. \( a^2 + 6a - 7 = (a\, \ldots\, ) (a\, \ldots\, ) \)
   \[= (a + 7)(a - 1) \]
   [the first term in each bracket must be 'a']
   [Find \( m \) and \( n \) so that \( m \times n = -7 \) and \( m + n = +6 \)
   \( \Rightarrow m = 7 \) and \( n = -1 \) since larger number must be +ve]

Some expressions may have irrational factors or no factors at all.
The discriminant can be used to check the nature of the factors of a quadratic expression

For a quadratic expression \( ax^2 + bx + c \), \( b^2 - 4ac \) is called the **discriminant**

- \( b^2 - 4ac > 0 \) and a perfect square \( \Rightarrow \) rational factors (can be factorised as above)
- \( b^2 - 4ac > 0 \) but not a perfect square \( \Rightarrow \) irrational factors (use quadratic formula)
  - \( b^2 - 4ac = 0 \Rightarrow x^2 + bx + c = (x + m)^2 \)
- \( b^2 - 4ac < 0 \Rightarrow \) there are no real factors

Example

*Factorise (if possible) \( a^2 + 6a + 12 \)*

\( a = 1, \ b = 6, \ c = 12 \)

and \( b^2 - 4ac = 6^2 - 4 \times 1 \times 12 = -12 \)

The discriminant is negative, therefore \( a^2 + 6a + 12 \) has no real factors.

Exercise

Factorise the following (if possible).

1. \( x^2 + 10x + 21 \)
2. \( m^2 - m - 72 \)
3. \( x^2 - 2x - 24 \)
4. \( n^2 + 6n + 16 \)
5. \( y^2 + 20y + 51 \)
6. \( a^2 - 15a + 44 \)

Answers

1. \((x + 7)(x + 3)\) 2. \((m - 9)(m + 8)\) 3. \((x - 6)(x + 4)\) 4. no real factors 5. \((y + 17)(y + 3)\) 6. \((a - 11)(a - 4)\)
FACTORISATION: PERFECT SQUARES

Perfect squares

Examples of perfect squares are $5^2$, $x^2$, $a^2$, $b^2$, $(xy)^2$ and $(a + b)^2$

$(a + b)^2$ can be expanded as follows

$(a + b)^2 = a^2 + 2ab + b^2$

Similarly $(a - b)^2 = a^2 - 2ab + b^2$

(a + b)$^2$ does not equal $a^2 + b^2$ and (a - b)$^2$ does not equal $a^2 - b^2$

Note that the expansion gives three terms:

- the first and last terms of the expansion must be positive
- the middle term is twice the product of the first and last terms
- the middle term may be positive or negative

The rule for expanding perfect squares can be used in reverse for factorising perfect squares.

Examples

Check each of the following expressions and, if it is a perfect square, state the perfect square.

1. $x^2 + 14x + 49$  
   $\quad= (x + 7)^2$
   - first term is the square of $x$
   - last term is positive and is the square of $7$
   - middle term $= 2 \times x \times (\pm 7)$

2. $y^2 - 20y + 25$
   - first term is the square of $y$
   - last term is positive and is the square of $\pm 5$
   - BUT middle term does not equal $2 \times y \times (\pm 5)$

3. $4a^2 - 12a - 9$
   - first term is the square of $2a$
   - last term is negative ($9$ is a perfect square but $-9$ is not)
   - first term is $(10x)^2$
   - last term is $(\pm 9)^2$
   - middle term is $2 \times 10x \times (\pm 9)$

4. $100x^2 - 180x + 81$
   - Not a perfect square but can be factorised
   - And $25x^2 + 40x + 16$ is a perfect square

5. $50x^2 + 80x + 32$
   $\quad= 2(25x^2 + 40x + 16)$
   $\quad= 2(5x + 4)^2$

Exercise

Check each of the following expressions. If it is a perfect square factorise.

1. $a^2 + 2a + 1$
2. $x^2 - 4x + 4$
3. $25x^2 - 10x + 1$
4. $81x^2 + 108x + 36$
5. $9a^2 - 24a - 16$
6. $2x^2 + 8x + 8$

Answers

1. $(a + 1)^2$
2. $(x - 2)^2$
3. $(5x - 1)^2$
4. $(9x + 6)$
5. Not a perfect square
6. $2(x + 2)^2$
FACTORISATION: DIFFERENCE OF TWO SQUARES

Consider the following expansion:

\[(x + 5)(x - 5) = x^2 + 5x - 5x + 25\]
\[= x^2 - 25\]

Note that:

- the terms in the brackets differ only in the sign of the second term
- the expansion is the difference of two terms, both of which are perfect squares

In general:

\[(a + b)(a - b) = a^2 - b^2\]

This can be used to factorise expressions of the form \(a^2 - b^2\)

**Examples**

1. \[a^2 - 36 = (a)^2 - (6)^2\]
   \[= (a + 6)(a - 6)\]  \[\text{[expression is the difference of two squares]}\]
   \[\text{['DOTS’ rule (order of the terms on right hand side is not important)']}\]

2. \[4x^2 - y^2 = (2x)^2 - (y)^2\]
   \[= (2x + y)(2x - y)\]  \[\text{['DOTS’ rule']}\]

3. \[3x^2 - 48 = 3(x^2 - 16)\]
   \[= 3[(x)^2 - (4)^2]\]
   \[= 3(x + 4)(x - 4)\]  \[\text{[not yet a difference of two squares so first take out a common factor of 3]}\]

4. \[(x + 2)^2 - 9 = (x + 2)^2 - (3)^2\]
   \[= (x + 2 + 3)(x + 2 - 3)\]
   \[= (x + 5)(x - 1)\]  \[\text{['DOTS’ rule']}
   \[\text{[Simplify]}\]

5. \[y^2 + 36\]  \[\text{[this is not the difference of two squares and has no real factors]}\]

**Exercise**

Factorise the following using the ‘DOTS’ rule (if possible).

1. \[x^2 - 4\]
2. \[49 + x^2\]
3. \[121x^2 - 49y^2\]
4. \[a^2b^2 - 25\]
5. \[5x^2 - 20\]
6. \[x^2y^3 - 36y\]
7. \[(x + 5)^2 - 36\]
8. \[(a + 1)^2 - (b - 2)^2\]

**Answers**

1. \[(x + 2)(x - 2)\]
2. \[\text{Does not factorise}\]
3. \[(11x + 7y)(11x - 7y)\]
4. \[(ab + 5)(ab - 5)\]
5. \[5(x + 2)(x - 2)\]
6. \[y(xy + 6)(xy - 6)\]
7. \[(x + 11)(x - 1)\]
8. \[(a + b - 1)(a - b + 3)\]
FACTORISATION: QUADRATICS (ii)

Consider the expansion of the product \((3x + 2)(2x + 1)\):

\[(3x + 2)(2x + 1) = 6x^2 + 3x + 4x + 2\]

\[= 6x^2 + 7x + 2\]

- Multiplying the first term in each bracket gives the '6x^2' term
- Multiplying the last term in each bracket gives the constant term \((+2)\),
- The coefficient of the 'x' term is the sum of the 'x' terms: \((+3x + 4x = +7x)\).

Factorisation of \(ax^2 + bx + c\) with \(a \neq 1\)

Technique similar to those used for expressions of the type \(x^2 + bx + c\) are suitable but in this case the coefficient of \(x\), in at least one bracket, will not equal 1.

Examples

1. Factorise \(2x^2 + 7x + 6\)

\[2x^2 + 7x + 6 = (2x\ldots)(x\ldots)\]

\[= (2x + 3)(x + 2)\]

[2x \times x = 2x^2 and we need to choose factors of +6]

[middle term +ve, try all combinations of +6 and +1 or +3 and +2]

[gives 3x + 4x = 7x for middle term]

2. Factorise \(2x^2 - x - 6\)

\[2x^2 - x - 6 = (2x\ldots)(x\ldots)\]

\[= (2x + 3)(x - 2)\]

[2x \times x = 2x^2 and we need to choose factors of -6]

[middle term -ve, try all combinations of ±6 and ±1 or ±3 and ±2]

[gives 3x - 4x = -x for middle term]

3. Factorise \(3a^2 - 16a + 5\)

\[3a^2 - 16a + 5 = (3a\ldots)(a\ldots)\]

\[= (a - 5)(3a - 1)\]

[3a \times a = 3a^2 and we need to choose factors of +5]

[middle term -ve, so -5 and -1 are the correct factors]

[gives -15a - a = 16a for middle term]

4. Factorise \(6y^2 + 16y - 6\)

\[6y^2 + 16y - 6 = 2(3y^2 + 8y - 3)\]

\[= 2(3y\ldots)(y\ldots)\]

\[= 2(3y - 1)(y + 3)\]

[Remove a common factor of 2]

[3y \times y = 3y^2 and we need to choose factors of -6]

[middle term -ve, try all combinations of ±6 and ±1 or ±9 and ±1]

[gives +9y - y = 8y for middle term]

Exercise

Factorise (if possible) the following

(1) \(5x^2 + 13x + 6\)
(2) \(2x^2 + x - 15\)
(3) \(3m^2 - m - 2\)
(4) \(3y^2 - 10y + 8\)
(5) \(2a^2 + 11a + 12\)
(6) \(6x^2 - 11x + 5\)

Answers

(1) \((5x + 3)(x + 2)\)
(2) \((2x - 5)(x + 3)\)
(3) \((3m + 2)(m - 1)\)
(4) \((3y - 4)(y - 2)\)
(5) \((2a + 3)(a + 4)\)
(6) \((6x - 5)(x - 1)\)
FACTORISATION: COMPLETING THE SQUARE

A quadratic expression may be factorised by the method of “completing the square” even when the factors are not rational.

A **perfect square** is of the form \((x + b)^2 = x^2 + 2bx + b^2\)

The first two terms of the right-hand side of the above equation are \((x^2 + 2bx\). To make a perfect square or to **complete the square** on \(x^2 + 2bx\), the third term, \(b^2\) must be added.

Note that \(b^2 = (\text{half the coefficient of } x)^2\).

**Examples**

1) **Complete the square on** \(x^2 + 6x\)

\[
x^2 + 6x\text{ are the first two terms of the complete square}
\]

\[
= (x + 3)^2 \quad \text{[half of 6]}^2 = 3^2 = 9
\]

2) **Complete the square on** \(x^2 + 5x\)

\[
x^2 + 5x\text{ are the first two terms of the complete square}
\]

\[
= (x + \frac{5}{2})^2 \quad \text{[half of 5]}^2 = \frac{5}{2}^2 = \frac{25}{4}
\]

3) **Complete the square on** \(x^2 - 10x\)

\[
x^2 - 10x\text{ are the first two terms of the complete square}
\]

\[
= (x - 5)^2 \quad \text{[half of 10]}^2 = 5^2 = 25
\]

**Factorisation**

Expressions of the form \(x^2 + bx + c\) can sometimes be factorised by completing the square on the \(x^2 + bx\) and rewriting \(x^2 + bx + c = (x + \frac{b}{2})^2 - \left(\frac{b}{2}\right)^2 + c\)

\[
= (x + \frac{b}{2})^2 - \left(\frac{b}{2}\right)^2 - c
\]

\[
= (x + \frac{b}{2})^2 - m^2 \quad \text{where } m^2 = \left(\frac{b}{2}\right)^2 - c
\]

The DOTS rule may then be used to factorise the expression.

**Examples**

Factorise by the method of completing the square

1) \(x^2 + 8x - 5 = (x^2 + 8x + 16) - 16 - 5\)

\[
= (x + 4)^2 - 21
\]

\[
= (x + 4)^2 - \sqrt{21}^2
\]

\[
= (x + 4 + \sqrt{21})(x + 4 - \sqrt{21})
\]

[Halve the coefficient of x and square: \((8/2) = 4\) and \(4^2 = 16\)]
[Add and subtract 16].

[Write as the difference of two squares].

[Factorise using ‘DOTS’]

2) \(x^2 - 6x - 4 = (x^2 - 6x + 9) - 9 - 4\)

\[
= (x - 3)^2 - 13
\]

\[
= (x - 3)^2 - \sqrt{13}^2
\]

\[
= (x - 3 + \sqrt{13})(x - 3 - \sqrt{13})
\]

[Halve the coefficient of x and square: \((6/2) = 3\) and \(3^2 = 9\)]
[Add and subtract 9].

[Write as the difference of two squares].

[Factorise using ‘DOTS’]
If the coefficient of $x^2$ is not 1, remove the coefficient as a common factor and then factorise by completing the square.

3) $2x^2 - 10x + 2 = 2(x^2 - 5x + 1)$

   $= 2(x^2 - 5x + (\frac{5}{2})^2 - (\frac{5}{2})^2 + 1)$

   $= 2[(x - \frac{5}{2})^2 - \frac{25}{4} + 1]$  

   $= 2[(x - \frac{5}{2})^2 - \frac{21}{4}]$  

   $= 2[(x - \frac{5}{2})^2 - (\frac{\sqrt{21}}{2})^2]$  

   $= 2[(x - \frac{5}{2} + \frac{\sqrt{21}}{2})(x - \frac{5}{2} - \frac{\sqrt{21}}{2})]$  

Exercise

Factorise by completing the square (if possible).

(1) $x^2 + 6x - 5$       (2) $a^2 - 8a - 2$       (3) $m^2 + 2m + 10$       (4) $4x^2 + 8x - 20$

Answers

(1) $(x + 3 + \sqrt{14})(x + 3 - \sqrt{14})$  
(2) $(a - 4 + \sqrt{18})(a - 4 - \sqrt{18})$  
(3) no real factors  
(4) $4(x + 1 + \sqrt{5})(x + 1 - \sqrt{5})$