RMIT UNIVERSITY

A3.5 FACTORISATION: COMPLETING THE SQUARE

A quadratic expression may be factorised by the method of "completing the square".

A **perfect square** is of the form $(x + b)^2 = x^2 + 2bx + b^2$

The first two terms of the right-hand side of the above equation are ($x^2 + 2bx$). To make a perfect square or to **complete the square** on $x^2 + 2bx$,the third term, b^2 must be added.

Note that b^2 = (half the coefficient of x)^2.

Examples

To complete the square on $x^2 + 6x$ add (half of 6)^2, $= 3^2 = 9$

$x^2 + 6x$ are the first two terms of the complete square $x^2 + 6x + 9$

To complete the square on $x^2 + 5x$ add (half of 5)^2, $= \left(\frac{5}{2}\right)^2$

$x^2 + 5x$ are the first two terms of the complete square $x^2 + 5x + \left(\frac{5}{2}\right)^2$

To complete the square on $x^2 - 10x$ add (half of 10)^2, $= 5^2 = 25$

$x^2 - 10x$ are the first two terms of the complete square $x^2 - 10x + 25$

Expressions of the form $x^2 + bx + c$ can be factorised by completing the square on the first two terms and rewriting the expression in the form

$$\left[\left(x + \frac{b}{2}\right)^2 - m^2\right] \text{where } m^2 = -\left(\frac{b}{2}\right)^2 + c$$

The DOTS rule can then be used to factorise the expression.

Examples

Factorise by the method of completing the square

1. x^2+8x-5:

Halve the coefficient of x (8/2) =4
Square the result 4^2 =16

Add and subtract 16. (This does not change the expression)

$$x^2+8x-5=\left(x^2+8x+16\right)-16-5$$

Simplify
$x^2+8x+16=(x+4)^2$
– 16 – 5 = – 21

$$=\left(x+4\right)^2-21$$

Write as the difference of two squares.

$$=\left(x+4\right)^2-\left(\sqrt{21}\right)^2$$

factorise using 'DOTS'

$$x^2+8x-5=\left(x+4+\sqrt{21}\right)\left(x+4-\sqrt{21}\right)$$

2. x^2-6x-4

Halve the coefficient of x. (6/2) = 3
Square the result 3^2 =9
Add and subtract 9
Simplify
$x^2-6x+9=(x-3)^2$, – 16 – 4 = – 21

$$x^2-6x-4=\left(x^2-6x+9\right)-9-4$$

$$=\left(x-3\right)^2-13$$

Write as the difference of two squares.

$$=\left(x-3\right)^2-\left(\sqrt{13}\right)^2$$

factorise using 'DOTS'

$$x^2-6x-4=\left(x-3+\sqrt{13}\right)\left(x-3-\sqrt{13}\right)$$

See Exercise 1

Examples where the coefficient of x^2 is not 1

If the coefficient of x^2 is not 1, remove the coefficient as a common factor and then factorise by completing the square.

3. $2x^2-10x+2$

Remove the common factor of 2

$$2x^2-10x+2=2(x^2-5x+1)$$

Halve the coefficient of x, Square the result, then add and subtract the answer. $+\left(\frac{5}{2}\right)^2-\left(\frac{5}{2}\right)^2$

$$=2\left[\left(x^2-5x+\left(\frac{5}{2}\right)^2-\left(\frac{5}{2}\right)^2+1\right)\right]$$

$$=2\left[\left(x-\frac{5}{2}\right)^2-\frac{25}{4}+\frac{4}{4}\right]$$

$$= 2\left[\left(x - \frac{5}{2}\right)^2 - \frac{21}{4}\right]$$ simplify

$$= 2\left[\left(x - \frac{5}{2}\right)^2 - \left(\frac{\sqrt{21}}{2}\right)^2\right]$$ write as difference of two squares

$$= 2\left[\left(x - \frac{5}{2} + \frac{\sqrt{21}}{2}\right)\left(x - \frac{5}{2} - \frac{\sqrt{21}}{2}\right)\right]$$ factorise using DOTS

See Exercise 2

Exercise 1

Factorise by completing the square (if possible).

(a) $x^2 + 6x - 5$ (b) $x^2 + 4x + 2$ (c) $a^2 - 8a - 2$ (d) $x^2 + 6x + 10$

(e) $x^2 - 5x + 5$ (f) $y^2 + 3y + 4$ (g) $a^2 - 5a - 1$

Exercise 2

(a) $4x^2 + 8x - 20$ (b) $x^2 - 6x + 2$ (c) $12 - 4x - 2x^2$

Answers

Exercise 1

(a) $(x + 3 + \sqrt{14})(x + 3 - \sqrt{14})$

(b) $(x + 2 + \sqrt{2})(x + 2 - \sqrt{2})$

(c) $(a - 4 + 3\sqrt{2})(a - 4 - 3\sqrt{2})$

(d) no real factors

(e) $(x - \frac{5}{2} + \frac{\sqrt{5}}{2})\ (x - \frac{5}{2} - \frac{\sqrt{5}}{2})$

(f) no real factors

(g) $(a - \frac{5}{2} + \frac{\sqrt{29}}{2})\ (x - \frac{5}{2} - \frac{\sqrt{29}}{2})$

Exercise 2

(a) $4(x + 1 + \sqrt{6})(x + 1 - \sqrt{6})$

(b) $(x - 3 + \sqrt{7})(x - 3 - \sqrt{7})$

(c) $-2(x + 1 + \sqrt{7})(x + 1 - \sqrt{7})$