A3.4 FACTORISATION: QUADRATICS

Definition

The general form of a quadratic expression is: $ax^2 + bx + c \quad a \neq 0$

where *a*, *b*, *c* are constants and *x* is the unknown.

Expansion

To expand an expression of the form $(a+b)(c+d)$, multiply each term in the first bracket by each term in the second bracket.

$$(x+2)(x+3) = x(x+3) + 2(x+3) = x^2 + 3x + 2x + 6$$

Simplify by combining like terms :

$$(x+2)(x+3) = x^2 + 5x + 6$$

Factorisation

Factorisation is the reverse of expansion.

$x^2 + 5x + 6$ is expressed as the product of two factors, $(x+2)$ and $(x+3)$.

factorisation →

$$x^2 + 5x + 6 = (x+2)(x+3)$$

← expansion

Note that:

- multiplying the first term in each bracket gives the term 'x^2' in the expression
- multiplying the last term in each bracket gives the constant term, (+6), in the expression
- the coefficient of the 'x' term is the sum of last term in each bracket (+2 + 3 = +5).

In general:

constant 'c' positive: $x^2 + bx + c = (x+m)(x+n)$ and $x^2 - bx + c = (x-m)(x-n)$

constant 'c' negative: $x^2 + bx - c = (x+m)(x-n) \quad m > n$

$x^2 - bx - c = (x+m)(x-n) \quad m < n$ where m and n are ≥ 0.

Expressions with $a = 1$

If $a = 1$ the expression becomes $\boldsymbol{x^2 + bx + c}$

To factorise:

Find two numbers that: • multiply to '*c*' and • add to '*b*' Insert one number into each of the brackets (x ….)(x ….). The order is not important.

Note:

- If 'c' is positive, both numbers are positive or both numbers are negative.

  If b is positive, both numbers are positive.

  If b is negative, both numbers are negative.

- If c is negative, then one number is positive and the other negative;

  If b is positive, then the larger number is positive.

  If b is negative, then the larger number is negative.

Examples

Factorise the following.

1. $x^2 + 9x + 14$

$x^2 + 9x + 14 = (x\ldots\ \ )(x\ldots\ )$

$= (x + 2)(x + 7)$

the first term in each bracket must be 'x'

Look for two numbers that

- multiply to $+14$
- add to $+9$

the numbers are $+2$ and $+7$

2. $y^2 - 7y + 12$

$y^2 - 7y + 12 = (y\ldots.)(y\ldots\ )$

$= (y - 3)(y - 4)$

the first term in each bracket must be 'y'

Look for two numbers that

- multiply to $+12$.
- add to -7,

the numbers are -3 and -4

3. $p^2 - 5p - 14$

$p^2 - 5p - 14 = (p\ldots\ \ )(p\ldots.\ )$

$= (p - 7)(p + 2)$

the first term in each bracket must be 'p'

Look for two numbers that

- multiply to -14
- add to -5,

(the larger number must be $-$ve)

numbers are -7 and $+2$

4. $a^2 + 6a - 7$

$a^2 + 6a - 7 = (a\ldots \quad)(a\ldots \quad)$

$= (a + 7)(a - 1)$

the first term in each bracket must be 'a'

Look for two numbers that

- multiply to -7
- add to $+6$

(the larger number must be $+$ve)

numbers are $+7$ and -1

Not all expressions of the form $x^2 + bx + c$ can be factorised by this method. The factors may be of a more complicated form or there may be no real factors.

Discriminant

For a quadratic equation of the form $ax^2 + bx + c$ a method of checking for real factors is to calculate the value of ***(b² – 4ac)*** (called the ***discriminant***).

If $b^2 - 4ac < 0$ there are no real factors

Example

Factorise (if possible) $a^2 + 6a + 12$

In this case:

'b' $= +6$ 'c' $= +12$

$\therefore$ $(b^2 - 4ac) = -12$

The discriminant is negative therefore $a^2 + 6a + 12$ has no real factors.

Exercise 1

Factorise the following (if possible).

1. $x^2+10x+21$	**2.** $z^2+11z+18$	**3.** $x^2+5x-14$
4. m^2-m-72	**5.** x^2+6x+9	**6.** $a^2-15a+44$
7. $x^2-2x-24$	**8.** $y^2-10y+16$	**9.** $z^2+4z-60$
10. $n^2+6n-16$	**11.** $a^2+5a+10$	**12.** $s^2+2s-48$
13. $y^2+7y+19$	**14.** $x^2+16x+39$	**15.** $x^2-14x+45$

Answers

1. $(x+7)(x+3)$	2. $(z+9)(z+2)$	3. $(x+7)(x-2)$
4. $(m-9)(m+8)$	5. $(x+3)(x+3)$	6. $(a-11)(a-4)$
7. $(x-6)(x+4)$	8. $(y-2)(y-8)$	9. $(z-6)(z+10)$
10. $(n-2)(n+8)$	11. no real factors	12. $(s+8)(s-6)$
13. no real factors	14. $(x+13)(x+3)$	15. $(x-9)(x-5)$

Expressions with a ≠ 1

Expressions of the type $ax^2 + bx + c$ can be factorised using a technique similar to that used for expressions of the type $x^2 + bx + c$.

In this case the coefficient of x, in at least one bracket, will not equal 1.

Consider the following product.

$$\begin{aligned}(3x+2)(2x+1) &= 6x^2 + 3x + 4x + 2 \\ &= 6x^2 + 7x + 2\end{aligned}$$

- multiplying the first term in each bracket gives the 'x^2' term ($6x^2$) in the expansion
- multiplying the last term in each bracket gives the constant term (+2), in the expansion
- the coefficient of the 'x' term is the sum of the 'x' terms in the expansion. ($+3x + 4x = +7x$).

Examples

1. Factorise $2x^2 + 7x + 6$

$$2x^2 + 7x + 6 = (2x\ldots \quad)(x\ldots \quad)$$

$$2x^2 + 7x + 6 = (2x + 3)(x + 2)$$

- $2x \times x = 2x^2$
- find factors of + 6

  middle term positive so

  + 6 and +1 or +3 and +2
- try all combinations.
- the one that gives +7x when the brackets are expanded is $(2x + 3)$ and $(x + 2)$

2. Factorise $2x^2 - x - 6$

$$2x^2 - x - 6 = (2x\ldots \quad)(x\ldots \quad)$$

$$2x^2 - x - 6 = (2x + 3)(x - 2)$$

- $2x \times x = 2x^2$
- find factors of – 6.
- +6 & –1 or –6 & +1 or +3 & –2 or –3 & +2
- try all combinations
- the one that gives – x when the brackets are expanded is $(2x+3)$ and $(x - 2)$

3. Factorise $3a^2 - 16a + 5$

$$3a^2 - 16a + 5 = (3a\ldots \quad)(a\ldots \quad)$$

$$3a^2 - 16a + 5 = (a - 5)(3a - 1)$$

- $3a \times a = 3a^2$
- find factors of +5
- middle term negative so – 5 and – 1
- try each combination
- the one that gives – 16a when the brackets are expanded is $(a - 5)$ and $(3a - 1)$

4. Factorise $6y^2 + 16y - 6$

$6y^2 + 16y - 6 = 2(3y^2 + 8y - 3)$

$= 2(\,3y\ldots\quad)(y\ldots\quad)$

Remove a common factor of 2.

This reduces the number of possible combinations.

$6y^2 + 16y - 6 = 2(3y - 1)(y + 3)$

- $3y \times y = 3y^2$
- find factors of -3
- possibilities are $+3$ and -1 or -3 and $+1$
- try all combinations
- the one that gives $+8y$ when the brackets are expanded is $(3y - 1)$ and $(y + 3)$

To factorise a quadratic of the form $ax^2 + bx + c$

- search each term in the expression for a common factor (**every term** must have this factor).
- remove this from the expression then place before the brackets.
- factorise, if possible, the resulting quadratic expression

This approach is good provided the number of combinations of factors of 'a' and 'c' is limited.

For more complicated expressions of the type $ax^2 + bx + c$ there are other methods of factorisation which may be more suitable.

Exercise 2

Factorise the following

1. $5x^2 + 13x + 6$ **2.** $2x^2 + x - 15$ **3.** $3m^2 - m - 2$

4. $3y^2 - 10y + 8$ **5.** $2a^2 + 11a + 12$ **6.** $6x^2 - 11x + 5$

Answers

1. $(5x + 3)(x + 2)$ **2.** $(2x - 5)(x + 3)$ **3.** $(3m + 2)(m - 1)$

4. $(3y - 4)(y - 2)$ **5.** $(2a + 3)(a + 4)$ **6.** $(6x - 5)(x - 1)$