A1.7 PARTIAL FRACTIONS

Adding fractions

To add fractions, rewrite the fractions with a common denominator then add the numerators.

For example:

$$\frac{3}{3x+4} + \frac{2}{x-5} = \frac{3}{3x+4} \times \frac{x-5}{x-5} + \frac{2}{x-5} \times \frac{3x+4}{3x+4}$$ [a common denominator is $(3x + 4)(\,x - 5)$]

$$= \frac{3x-15}{(3x+4)(x-5)} + \frac{6x+8}{(3x+4)(x-5)}$$

$$= \frac{3x - 15 + 6x + 8}{(3x+4)(x-5)}$$

$$= \frac{9x-7}{(3x+4)(x-5)}$$

Finding Partial fractions

The reverse of this process is to split a fraction into partial fractions. In the above example

$$\frac{9x-7}{(3x+4)(x-5)} = \frac{3}{3x+4} + \frac{2}{x-5}$$

Algebraic fraction   Partial fractions

The first step in finding partial fractions is to factorise the denominator. The factorisation will determine the form of the partial fractions:

$\frac{mx+k}{(x-a)(x-b)}$	(distinct linear factors)	$\frac{A}{x-a} + \frac{B}{x-b}$	A, B constant
$\frac{mx+k}{(x-a)^2}$	(repeated linear factor)	$\frac{A}{x-a} + \frac{B}{(x-a)^2}$	A, B constant
$\frac{mx+k}{(ax^2+bx+c)(px+q)}$	(quadratic factor and linear factor)	$\frac{A}{px+q} + \frac{Bx+C}{ax^2+bx+c}$,	A, B, C constant

See Exercise 1

To express an algebraic fraction as partial fractions:

1. Factorise the denominator.

2. Write the algebraic fraction in partial fraction form with unknown constants as above

3. Add the partial fractions

4. Then equate coefficients , or substitute values of x, to determine the value of the constants

Examples

1. Express $\frac{x-5}{x^2+2x-3}$ as the sum of partial fractions

$$\frac{x-5}{x^2+2x-3} = \frac{x-5}{(x-1)(x+3)} \Rightarrow \frac{x-5}{x^2+2x-3} = \frac{A}{x-a} + \frac{B}{x-b} \quad \text{[distinct linear factors]}$$

$$\Rightarrow \frac{x-5}{x^2+2x-3} = \frac{A}{x-1} + \frac{B}{x+3}$$

$$\Rightarrow \frac{x-5}{x^2+2x-3} = \frac{A(x+3)+B(x-1)}{x^2+2x-3} \quad \text{[common denominator is } (x-1)(x+3)\text{]}$$

$$\Rightarrow \quad x-5 = A(x+3)+B(x-1) \quad \text{[Equating numerators]}$$

$$\Rightarrow \quad x-5 = Ax+3A+Bx-B \quad \text{[Expanding the brackets]}$$

$$\Rightarrow \quad x-5 = (A+B)x+(3A-B) \quad \text{[Collect like terms of } x\text{]}$$

Equating coefficients of x we find that

$A + B = 1$ and

$3A - B = -5$

Solving these simultaneous equations we find the $A = -1$ and $B = 2$

therefore $$\frac{x-5}{x^2+2x-3} = \frac{-1}{x-1} + \frac{2}{x+3}$$

Alternative method

Solving the simultaneous equations that result from equating coefficients can sometimes be quite lengthy. An alternative method is to equate the numerators and, before expanding the brackets, substitute a value of x into both sides of the equation so that only one variable remains. Repeat this to find other variables. This method will not necessarily find all variables, but will often make calculations easier.

In the above example, after the line: $x-5 = A(x+3)+B(x-1)$ we can substitute any convenient value for x

$$x-5 = A(x+3)+B(x-1)$$

If $x = -3$ then $-3 - 5 = A(0) + B(-3 - 1)$ $\Rightarrow -8 = -4B$

$\Rightarrow B = 2$

If $x = 1$ then $1 - 5 = A(1 + 3)) + B(0)$ $\Rightarrow -4 = 4A$

$\Rightarrow A = -1$

Using the method of substituting convenient values for x

$$\therefore \frac{x-5}{x^2+2x-3} = \frac{-1}{x-1} + \frac{2}{x+3}$$

This is the same answer as we achieved above, using simultaneous equations to solve for A and B.

2. Express $\frac{5x^2+3x+1}{x^3-3x-2}$ as the sum of partial fractions

$$\frac{5x^2+3x+1}{x^3-3x-2} = \frac{5x^2+3x+1}{(x+1)^2(x-2)}$$ [factorising the denominator]

$$\Rightarrow \quad \frac{5x^2+3x+1}{(x+1)^2(x-2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-2}$$ [repeated linear factors]

$$\Rightarrow \quad \frac{5x^2+3x+1}{(x+1)^2(x-2)} = \frac{A(x+1)(x-2)+B(x-2)+C(x+1)^2}{(x+1)^2(x-2)}$$ [finding a common denominator]

$$\Rightarrow \quad 5x^2 + 3x + 1 = A(x+1)(x-2) + B(x-2) + C(x+1)^2$$ [Equating numerators]

We can now substitute any convenient value for x

If $x = 2$ then $5(2)^2 + 3(2) + 1 = A(0) + B(0) + C(3)^2$

$\Rightarrow \quad 27 = 9C$

$\Rightarrow \quad C = 3$

If $x = -1$ then $5(-1)^2 + 3(-1) + 1 = A(0) + B(-1-2) + C(0)^2$

$\Rightarrow \quad 3 = -3B$

$\Rightarrow \quad B = -1$

If $x = 0$ then $5(0)^2 + 3(0) + 1 = A(1)(-2) + -1(-2) + 3(1)^2$ [using $B = -1$ and $C = 3$]

$\Rightarrow \quad 1 = -2A + 2 + 3$

$\Rightarrow \quad A = 2$

Therefore $$\frac{5x^2+3x+1}{(x+1)^2(x-2)} = \frac{2}{x+1} - \frac{1}{(x+1)^2} + \frac{3}{x-2}$$

3. Express $\frac{x^2-3x+16}{x^3-5x^2+x-5}$ as the sum of partial fractions

$$\frac{x^2-3x+16}{x^3-5x^2+x-5} = \frac{x^2-3x+16}{(x^2+1)(x-5)}$$

$$\frac{x^2-3x+16}{x^3-5x^2+x-5} = \frac{A}{x-5} + \frac{Bx+C}{x^2+1}$$ [linear and quadratic factors]

$$\frac{x^2-3x+16}{x^3-5x^2+x-5} = \frac{A(x^2+1)+(Bx+C)(x-5)}{x^3-5x^2+x-5}$$ [common denominator is $x^3 - 5x^2 + x - 5$]

$$x^2 - 3x + 16 = A(x^2+1) + (Bx+C)(x-5)$$ [equating numerators]

$$x^2 - 3x + 16 = (A+B)x^2 + (C-5B)x + (A-5C)$$ [removing brackets and regrouping]

∴
$A + B = 1$
$C - 5B = -3$
$A - 5C = 16$

Using the method of equating coefficients and $A = 1$, $B = 0$, $C = -3$ } by solving simultaneous equations

Therefore $\frac{x^2-3x+16}{x^3-5x^2+x-5} = \frac{1}{x-5} - \frac{3}{x^2+1}$

The process of finding partial fractions can only be performed on fractions where the degree of the numerator of the algebraic fraction is greater than that of the denominator. If necessary, divide the denominator into the numerator then express the remaining fractional part as partial fractions.

For example $\frac{x^2+7x+7}{(x+1)(x+2)} = 1 + \frac{4x+5}{(x+1)(x+2)} = 1 + \frac{1}{x+1} + \frac{3}{x+2}$ [after expressing as partial fractions]

See Exercise 2

Exercise 1

Rewrite each of the following in the appropriate generalized (do not calculate the constants) partial fractions form.

a) $\frac{x+6}{2x^2+5x-12}$ b) $\frac{2x}{(x^2+3)(x+1)}$ c) $\frac{2x}{x^2+8x+16}$ d) $\frac{x^2}{(x-1)(x+1)}$

Exercise 2

Express the following as partial fractions

a) $\frac{x+2}{x^2-5x+6}$ b) $\frac{3}{x^2-2x+1}$ c) $\frac{x^2-2x+2}{x^3+x^2+x}$ d) $\frac{x^2}{x^2-4}$

Answers

Exercise 1

a) $\frac{A}{2x-3} + \frac{B}{x+4}$ b) $\frac{Ax+B}{x^2+3} + \frac{C}{x+1}$ c) $\frac{A}{x+4} + \frac{B}{(x+4)^2}$ d) $1 + \frac{A}{x+1} + \frac{B}{x-1}$

Exercise 2

a) $\frac{5}{x-3} - \frac{4}{x-2}$ b) $\frac{3}{(x-1)^2}$ c) $\frac{2}{x} - \frac{x+4}{x^2+x+1}$ d) $1 - \frac{1}{x+2} + \frac{1}{x-2}$