RMIT UNIVERSITY

CN3 DE MOIVRE'S THEOREM

Integral Powers of Complex Numbers

De Moivre's theorem states that:

$$\text{If} \quad z = r cis\theta$$
$$\text{then} \quad z^n = r^n cis(n\theta)$$

Examples: *1. Express* $(1-i)^6$ *in the form* $x + yi$

$$(1-i)^6 = \left[\sqrt{2}cis\left(\frac{-\pi}{4}\right)\right]^6 \qquad \text{change to polar form}$$

$$= \left(\sqrt{2}\right)^6\left[cis\left(6\times\frac{-\pi}{4}\right)\right]^6 \qquad \text{by DeMoivre's theorem}$$

$$= 8cis\left(\frac{-3\pi}{2}\right)$$

$$= 8i$$

2. Simplify $\dfrac{\left(\sqrt{3}-i\right)^6}{(1+i)^8}$ *and give the answer in rectangular form*

$$\sqrt{3}-i = 2\,cis\left(-\frac{\pi}{6}\right) \qquad \text{change to polar form}$$

$$(\sqrt{3}-i)^6 = 64\,cis(-\pi) \qquad \text{by DeMoivre's theorem}$$

and $$1+i = \sqrt{2}cis\frac{\pi}{4} \qquad \text{change to polar form}$$

$$(1+i)^8 = 16cis2\pi \qquad \text{by De Moivre's theorem}$$

$$\therefore \quad \frac{\left(\sqrt{3}-i\right)^6}{(1+i)^8} = \frac{64cis(-\pi)}{16cis2\pi}$$

$$= \frac{64}{16}cis(-\pi-2\pi)$$

$$= 4\,cis(-3\pi)$$

$$= -4$$

See Exercise 1

Roots of a Complex Number

$z^n = rcis\theta$ will have n solutions of the form

$$z^{\frac{1}{n}} = r^{\frac{1}{n}} cis\left(\frac{\theta + 2\pi k}{n}\right), k = 0,1,.....n-1$$

Example: *Solve* $z^4 = 1 - \sqrt{3}i$

$1 - \sqrt{3}i = 2cis\left(-\frac{\pi}{3}\right)$ [change to polar form]

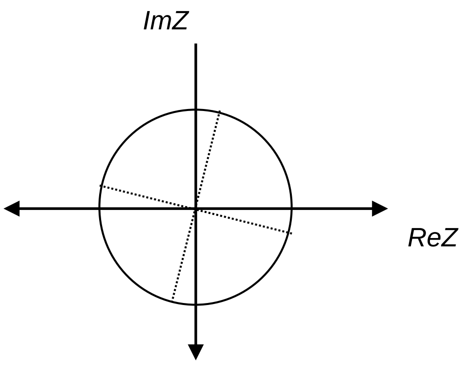
then $z^4 = 2cis\left(-\frac{\pi}{3}\right),\ 2cis\left(-\frac{\pi}{3}+2\pi\right),\ 2cis\left(-\frac{\pi}{3}+4\pi\right),\ 2cis\left(-\frac{\pi}{3}+6\pi\right)$

[as four solutions are required]

ie $z^4 = 2cis\left(-\frac{\pi}{3}\right),\ 2cis\left(\frac{5\pi}{3}\right),\ 2cis\left(\frac{11\pi}{3}\right),\ 2cis\left(\frac{17\pi}{3}\right)$

$\therefore\ z = 2^{\frac{1}{4}}cis\left(-\frac{\pi}{12}\right),\ 2^{\frac{1}{4}}cis\left(\frac{5\pi}{12}\right),\ 2^{\frac{1}{4}}cis\left(\frac{11\pi}{12}\right),\ 2^{\frac{1}{4}}cis\left(\frac{17\pi}{12}\right)$

The solutions may be represented graphically:



NB: The solutions of $z^n = rcis\theta$ lie on a circle with centre the origin and radius $r^{\frac{1}{n}}$ and they divide the circle into arcs of equal length. The symmetrical nature of the solutions can be used to find all solutions if one is known.

See Exercise 2

Exercises

Exercise 1

1. Evaluate giving your answers in polar form with $-\pi \le \theta \le \pi$

   (a) $\left(\sqrt{3}+i\right)^3$ (b) $(1-i)^5$ (c) $\left(-2\sqrt{3}+2i\right)^2$

2. Simplify each of the following giving the answer in polar form

   (a) $(1+i)^4(2-2i)^3$ (b) $\dfrac{\left(2-2\sqrt{3}i\right)^4}{(-1+i)^6}$

Exercise 2

1. Solve giving the answers in polar form with $-\pi \le \theta \le \pi$

   (a) $z^3 = -1$ (b) $z^4 = 16i$ (c) $z^3 = \sqrt{6}-\sqrt{2}i$

2. If $\sqrt{3}+i$ is one solution of $z^3 = 8i$ use a diagram to find the other solutions in rectangular form.

Answers

Exercise 1

1. (a) $8cis\dfrac{\pi}{2}$ (b) $2^{\frac{5}{2}}cis\dfrac{3\pi}{4}$ (c) $16cis\dfrac{-\pi}{3}$

2. (a) $2^{\frac{13}{2}}cis\left(\dfrac{\pi}{4}\right)$ (b) $32cis\left(\dfrac{\pi}{6}\right)$

Exercise 2

1. (a) $cis\left(\dfrac{\pi}{3}\right)$, $cis(\pi)$, $cis\left(\dfrac{-\pi}{3}\right)$ (b) $2cis\left(\dfrac{\pi}{8}\right), 2cis\left(\dfrac{5\pi}{8}\right), 2cis\left(\dfrac{-7\pi}{8}\right), 2cis\left(\dfrac{-3\pi}{8}\right)$

   (c) $\sqrt{2}cis\left(-\dfrac{\pi}{18}\right), \sqrt{2}cis\left(\dfrac{11\pi}{18}\right), \sqrt{2}cis\left(\dfrac{-13\pi}{18}\right)$

2. $-\sqrt{3}+i$ and -2i